IJCRT.ORG ISSN : 2320-2882
INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

SPECTRAL BEHAVIOUR AND SECOND ORDER DIFFERENTIAL EQUATION

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ABSTRACT

In this present paper we study about eigen functions expensions for a pair of second order differential equation. The theory of eigenfunction expansions associated with the second order differential equations and its spectral behavior.

Key Words: Matrix differential operator, convergence theorem, bounded variation.

1. Introduction

Hilbert took up the discussions on a pair of simultaneous differential equations of the second order and Whyburn, Kamke,. Lidskii, Levin, Kodaira, Coddinton and Levinson, Chakravarty, Bhagat (1-3), Tiwari studied problems with two (or more) simultaneous second order differential equations and advanced the theory to a large extent*- In what follows we sketch in brief only a few previous works on Spectra of differential equations.

2. Spectra of differential equations

The spectral theory of eigenfunction expansion associated with second-order differential equations goes at least as far baok as the time of Weyl. For definition of spectrum see for example Titchmarah [(8), p. 58]

3. Second-order differential equations

Wely used the method of integral equation in his discussions, but in the the forties Ticharah used the theory of complex variable methods to generalize some of Weyl's results. He made a systematic study of the theory of eigenfunction expansions till his death in early 1963.

Titchmarah's contributions are contained in his two volumes of Eigenfunction expansions, the fist being published in 1946. In this volume he considers the system (I) i.e.

$$\frac{d^2y}{dx^2} + (\lambda - q(x))y = 0 \qquad \text{(I)}$$

with certain boundary conditions and studies among others the nature of the spectrum depending on a variety of conditions satisfied by q(x). He proves that if.

(i) q(x) tend steadily to infinity so that $q'(x) \geq 0$

(ii) $q'(x) = 0[\{q(x)\}^c]$, $(o < c < \frac{3}{2})$

(iii) q'' (x) is ultimately of one sign,

then the spectrum of the system (I) with certain initial conditions is discrete. Further, if

(i) $q(x) < 0$, $q'(x) < 0$

(ii) $q(x) \to -\infty$

(iii) q'' (x) is ultimately of one sign,

and if $\int^{\infty} |q(x)|^{1/2} dx$ be divergent, the there is a continuous spectrum over $(-\infty, \infty)$. If in addition to the condition (i) - (iii), $\int^{\infty} |q(x)|^{-1/2} dx$ be convergent, then there is a continuous spectrum in $(-\infty, 0)$ and a point spectrum in $(0, \infty)$.

Winter [1948] discusses the location of continuous spectra for the system

$$y'' + (\lambda + f(x))\ y = 0 \qquad (0 \leq x < \infty) \qquad \text{(XVII)}$$

Hartman [1949] taken up the system

$$(p\ r')' + (q + \lambda)\ x = 0 \quad (0 \leq t < \infty) \qquad \text{(XVIII)}$$

with a boundary condition

$$x(0) \cos\alpha + x'(0) \sin\alpha = 0 \quad (0 < \alpha < \pi) \qquad \text{(XVIIIa)}$$

such that p(t) is positive and q(t) is a real valued function of t and (XVIII) possesses a solution which $\in L^2$ [0, ∞) (i.e. we are in the limit-point case). If S(α) denote the spectrum of the eigenvalue problem (XVIII), (XVIIIa), then

'There are exactly n values of λ belonging to S(α) which satisfy λ' < λ< λ" or λ' ≤ λ< λ" according as (XVIII) is oscillatory or non-oscillatory for λ = λ". In particular, n =∞ if and only if an infinite set of values of belonging to S(α) satisfies λ' < λ< λ".

Putnam [1949] studies the spectrum of the system

$$y'' + (\lambda + q)\, y = 0 \qquad \text{(XIX)}$$

where q = q(x) is real valued continuous function on the half-line 0 < x < ∞, λ a parameter and (XIX) is in Grenzpunkfall in the terminology of Weyl, with a boundary condition.

$$y(0)\,\mathrm{Cos}\alpha + y'(0)\,\mathrm{Sin}\alpha = 0, \quad 0 < \alpha < \pi. \qquad \text{(XIXa)}$$

He prove the theorem

If $\int^{\infty} q^2(x)\,(x)\,dx < \infty$, the the differential equation (XIX)

is in the Grenzpunkfall and the half-life λ ≥ 0 belongs to the spectrum of the boundary con value problem determined by the differential system (XIX) with boundary condition (XIXa)'.

The spectral function k() appear to have occurred first in the works of Titohmarsh and incorporated in his book on eigenfunction expansions, Vol.I, published in 1946. Kodaira also worked on it. Yosida [1950] obtained a shorter and simpler form of the Titchmarsh-Kodaira formula for the density matrix (i.e. the spectral matrix) which plays a prominent role in the theory of spectra.

Freidricks [1950] deals with the spectrum of characteristic values (eigen-values) of an ordinary linear self-adjoint differential equation.

$$\frac{d}{dx}\left\{\frac{p(x)d\phi}{dx}\right\} - q(x)\phi + \lambda r(x)\phi = 0 \qquad \text{(XX)}$$

with p(x) > 0 and r(x) > 0 in the interior of the basic interval of x, finite or infinite. Under certain conditions on p(x), q(x), r(x) the discreteness of the spectrum below a specific value of λ or total discreteness is ensured.

Sears [1951] proves the discreteness of the spectrum of the system (I), viz., y" + (λ-q(x)) y = 0, q(x) real and continuous, under certain conditions.

Naimark [1952-54] in a series of papers investigates the nature of the spectrum of singular non-selfadjoint differential operators of the second order.

Putnam [1955] gives a necessary and sufficient condition for the existence of negative spectra for the system x" + (λ + f) x = 0.

Aronszajn [1957] considers problems associated with spectra of

$$(-p\,x') + (q - \xi)\,x = 0, \qquad 0 < t < \infty , \qquad \text{(XXI)}$$

under certain conditions on p (t) and q(t) .

McLeod [1965] gives a sufficient condition for S [0,∞), S being the continuous spectrum of a self-adjoint extension of -4 = q (x) in $L^2(R^n)$, q(x) a real valued function, x ϵ R^n with certain conditions on q(x). He considers small potential in the paper under reference and applies the result to large negative potentials in a subsequent paper.

Kato [1966] in his perturbation theory for linear operators studies linear transformations in Hibert space. The theory contains a large part of the theory of eigenfunction expansions associated with second order differential equations. He gives a typical perturbation theorem giving information about the spectrum of a transformation of the form T + A, where T is unbounded and A satisfies a bounded or a complete continuity condition with respect to T.

Weidman [1967] concerns himself with the spectra of self-adjointness realization of A in L^2(a,b; r) or the operator

$$L = r\,(x)\text{-}1\left\{\left(\frac{-d}{dx}\right)p(x)\left(\frac{d}{dx}\right) + q(x)\right\} \qquad \text{(XXII)}$$

a < x < b, where p(x) > 0, r(x) > 0 leading to a number of results for the spectrum σ, the essential spectrum σ_{ϵ}, the continuous spectrum σ_c and absolutely continuous spectrum σ_a of A.

Choudhuri & Everitt [1968] consider properties of the spectrum of differential operators derived from the differential expression.

$$L[\psi] = -(p\psi')' + q\psi$$

on the interval [0 , ∞] with certain condition on p,q. The linear differential equation

$$L[\psi] = (-p\psi')' + q\psi = \lambda\psi \ (\lambda=\mu+i\upsilon) \qquad \text{(XXIII)}$$

is considered with the initial conditions at x = 0, viz.,

$\phi(0,\lambda) = -\sin\alpha$ $\quad \phi'(0,\lambda) = \{-p(0)\}^{-1}\cos\alpha$

$\theta(0,\lambda) = \cos\alpha$ $\quad \theta'(0,\lambda) = \{-p(0)\}^{-1}\cos\alpha$

θ(0,λ), ϕ(0,λ) being the usual fundamental set of solutions of (XXIII) Denoting the operator T (its domain being D(T) by

$T : D(T) \rightarrow L^2(0,\infty)$, $\quad Tf = L(f)$ for all $f \in D(T)$

and writing T_α to indicate its dependence on α, they prove among others

"The differential operator T_α has a disorete spectrum of and only if $m_\alpha(\lambda)$ is a meromorphic function of where $m_\alpha(\lambda)$ is associated with the L^2 - solution $\psi(x,\lambda) = \theta(x,\lambda) + m_\alpha(\lambda)\,\phi(x,\lambda)$ of (XXIII), and α indicates the dependence of $m_\alpha(\lambda) = m(\lambda)$ on α.

Eastern [1969] obtains a generalized result concerning the size of the gaps in the spectrum of the system (I), viz., y" + (λ-q(x))y = 0 where q(x) is a real piecewise, continue periodic function.

Everitt [1972] deals with the spectra of a second-order linear differential equation with p-integrable co-efficients.

4. System of first order equations

Roos and Sangren (23), 1961 outline a treatment for the system

$x'_1 - (\lambda + q_1)\, x_2 = 0 \qquad (0 < t < \infty) \qquad$ (XXIV)

$x'_2 - (\lambda + q_2)\, x_1 = 0$

with boundary condition.

$\cos\beta\, x_1(0) + \sin\beta\, x_2(0) = 0 \qquad$ (XXIVa)

They establish some criteria for the existence of the continuous spectra and point spectra associated with the system.

In a series of papers on the relativistic problems involving the pair of simultaneous differential equations

$(W\text{-}V + mc^2)\, \psi_1(r) = 0\ \{\psi'_1(r) = l\, \psi_2(r)/r\} \qquad$ (XXV)

$(W - V - mc^2)\, \psi_1(r) = -c\ \{\psi'_1(r) = l\, \psi_1(r)/r\}$

in which the soalar product V is a function of r only and W is the eignevalue parameter and c is the velocity of light, Titochmarsh [(8), 1961-62] thrown considerable light on the nature of the spectrum associated with the system, In one case he shows that the spectrum is continuous.

over $-\infty < W < \infty$ if $V(r) \to \infty$ as $r \to \infty$.

kim [1971] considers the singular Hamiltonian linear vector differential system.

$u'(t) = A(t)\, u(t) + B(t)\, v(t)$

$v'(t) = c(t)\, u(t) - A^*(t)\, v(t) \qquad$ (XXVI)

($t \in I$), u(t) , v(t) are complex-valued vector functions is an open interval I and A,B are non complex-valued matrix functions on I. He finds Green's matrix, spectral matrix and the parseval equality following the method of conditions and Levinson.

5. System of second and higher order differential equations

Bhagat [(1-3), 1970] establishes a spectral theorem for the system

$u''(x) + p(x)\, u(x) + r(x)\, v(x) + \lambda u(x) = 0 \qquad$ (XXVII)

$v''(x) + q(x)\, v(x) + r(x)\, v(x) + \lambda v(x) = 0$

($0 \le x < \infty$), where p(x), q(x), r(x) are real valued and continuous in $0 \le x < \infty$, with two linearly independent boundary conditions at x =0,

$a_{j1}\, u(0) + a_{j2}\, u'(0) + a_{j3}\, v(0) + a_{j4}\, v'(0) = 0 \qquad (j=1,2)$

where $p(x), q(x), r(x) \in L\,[0,\infty)$ with certain conditions on the constants aji (I = 1,2,3,4 j = 1,2).

Tiwari [1971] studies on the eigenvalues problem associated with the operator

$$L = \begin{pmatrix} -\frac{d^2}{dx^2} + p(x) & r(x) \\ r(x) & -\frac{d^2}{dx^2} + q(x) \end{pmatrix}$$

an operator similar to that introduced by Chakravarty in his oxford D.Phil. Thesis (1963), Aslo his paper [1965] and is a special case of that introduced by Bhagat in his Ph.D. Thesis [(3), 1966]. He develops a transform theory based on the operator in lines similar to those of seara and Titchmarsh (8). He obtain among others spectral matrix, expansion formula and parseval formula involving spectral matrix.

coddinton and Levinson [1955] mentions the differential system

$L(x) \equiv p_0\, x^{(n)} + p_1\, x^{(n-1)} + \ldots\ p_n(x) = lx \qquad$ (XXVIII)

with the boundary condition

$M_1\, x^{(n-1)}(a) + \ldots + M_n\, r(a) + N_1\, x^{(n-1)}(b) + \ldots N_n\, x(b) = 0$

where x is a vector with r components. They obtain among others spectral matrix, parseval theorem and the expansion formula.

.Kaljabin [1973] states a necessary and sufficient conditions for the disoreteness of the spectrum of matrix differential operator of the form

$l(y) = (-1)^n\, (p(x)\, y^{(n)})^{(n)`} \qquad$ (XXIX)

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