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STUDY OF CONVEX SPACES AND TENSOR PRODUCTS

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ABSTRACT
In this paper we will make use of the convex spaces and its tensor products.
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One easily shows that the tensor product is characterized by the following universal property:

1. Proposition

The map $\theta : X \times Y \to X \otimes Y$, defined by $\theta(x, y) = x \otimes y$, is a bilinear map with the property that any bilinear map $X \times Y \to Z$ to a vector space Z is the composition of θ with a unique linear map $\psi : X \otimes Y \to Z$.

If X and Y are locally convex topological vector spaces, there are at least two interesting and useful ways of giving $X \otimes Y$ a corresponding locally convex topology. The most natural of these is the projective tensor product topology, which we describe below :

If p and q are continous seminorms on X and Y, respectively, we define the tensor product seminorm $p \otimes q$ onn $X \otimes Y$ as follows:

$$(p \otimes q)(u) = \inf\left\{\sum p(x_i)q(y_i): u = \sum x_i \otimes y_i\right\}$$

It follows easily that $p \otimes q$ is, indeed, a seminorm.[1-4] Furthermore, we have:

Lemma

For seminorms p and q on X and Y.

(1) $(p \otimes q)(x \otimes y) = p(x)q \; for \; all \; x \in X, y \in Y;$

(2) If U = $\{x \in X : p(x) < 1\} \; and \; V = \{y \in Y : q(y) < 1\}, then$

$$co\left(\theta\,(U \times V)\right) = \{u \in X \otimes Y : (p \otimes q)(u) < 1\}$$

Proof. From the definition, it is clear that

$(p \otimes q)(x \otimes y) \le p(x)q(y)$ for all $x \in X, y \in Y$

On the other hand, for a fixed $(x, y) \in X \times Y$, using the Hahn-Banach theorem we may choose linear functional f on X and g on Y such that $f(x) = p(x), g(y) = q(y)$ and $|f(x')| \le p(x'), |q(y')$ for all $x', y') \in X \times Y$. Then if $x \otimes y = \sum x, \otimes y_i$ is any representation of $x \otimes y$ as a sum of rank one tensors, we have

$$p(x)q(y) = f(x)g)y) = \sum f(x_i)g(y_i) \le \sum p(x_i)q(y_i)$$

Since $(p \otimes q)(x \otimes y)$ is the inf of the expressions on the right side of this inequality we have $p(x)q(y) \le (p \otimes q)(x \otimes y)$. The proves (1).

Certainly $co\big(\theta(U \times V)\big) \subset \{u \in X \otimes Y : (p \otimes q)(u) < 1\}$ since the latter is a convex set containing $\theta(U \times V)$. To prove the reverse containment, let u be an element of $X \otimes Y$ with $(p\otimes q)(u) < 1$. Then we can represent u as $u = \sum x_i \otimes y_i$ with

$$\sum p(x_i)q(y_i) = r^2 < 1$$

If we set $x_i' = rp(x_i)^{-1}x_i \; and \; y_i' = rq(y_i)^{-1}y_i$, then $p(x_i') = r = q(y_i')$. Thus, $x_i' \in U$ and $y_i' \in V$. Furthermore, if $t_i = r^{-2}p(x_i)q(y_i)$. then

$$u = \sum t_i(x_i' \otimes y_i') \text{ and } \sum t_i = 1$$

Thus, $u \in co(\theta(U \times V))$ and the proof of (2) is complete [5-7].

Definition

The topology on $X \otimes Y$ determined by the family of seminorms $p \otimes q$, as above, will be called the projective tensor product topology. We will denote $X \otimes Y$, endowed with this topology, by $X \otimes Y$.

If $f \in X^*$ and $g \in Y^*$ then we may define a linear functional $f \otimes g$ on $X \otimes Y$ by

$$(f \otimes g)\left(\sum x_i \otimes y_i\right) = \sum f(x_i)g(y_i)$$

One easily checks that this is well defined and linear [5-6].

2. Proposition

The projective tensor product topology is a Hausfdorff locally convex topology on $X \otimes Y$ with the following properties :

(1) the bilinear map $\theta : X \times Y \to X \otimes_\pi Y$ is continuous;

(2) $f \otimes g \in (X \otimes_\pi Y)^*$ for each $f \in X^*$ and $g \in Y^*$;

(3) A neighbourhood base for the topology at 0 in $X \otimes_\pi Y$ consists of sets of the form $co(\theta(U \times v))$ where U is a 0- neigbhourhood in X and V is a 0 neighbourhood in Y.

(4) any continuous bilinear map $X \times Y \to Z$ to a locally convex space Z factors as the composition of θ with a unique continuous linear map $X \otimes_\pi Y \to Z$;

Proof. Lemma 1(1) implies that each $(p \otimes q) o\, \theta$ is continuous at (0,0) and this implies that θ is continuous at (0,0) and, hence, is continuous everywhere.

The continuity of $f \otimes g$ for $f \in X^*$ follows from the fact that $|f|$ and $|g|$ are continuous seminorms on X and Y and $|(f \otimes g)(\sum x_i \otimes y_i)| \le \sum |f(x_i)||g(y_i)| \le \sum |f(x_i)||g(y_i)| \le |f| \otimes |g|(\sum x_i \otimes y_i)$. This proves (2).

The fact that the projective topology is Hausdorff follows from (2). In fact, if $u \in X \otimes Y$ then we may write $u = \sum x_i \otimes y_i$, where the set $\{x_i\}$ is linearly independent. Then we may choose $f \in X^*$ such that $f(x_i) \neq 0$ if and only if $i = 1$ and we may choose $g \in Y^*$ such that $g(x_i) \neq 0$. Then the element $f \otimes g \in (X \otimes Y)^*$ has the non-zero value $f(x_1)g(x_1)$ at u. Thus, $U = \{v \in X \otimes Y : |(f \otimes g)(v)| < f(x_1)g(x_1)\}$ is an open set containing 0 but not containing u.

Part (3) is an immediate consequence of Lemma 1.(2)

If $\phi : X \times Y \to Z$ is a continuous bilinear map, then $\phi = \psi o \theta$ for a unique linear map $\psi : X \otimes_\pi Y \to Z$ by Proposition 2. To prove (4) we must show that ψ is continuous. Let W be a convex 0-neighbourhood in Z. Since ϕ is continuous, there exist 0-neighbourhoods U and V in X and Y, respectively, such that $\phi(U \times V) \subset W$. Then the convex hull of $\theta(U \times V)$ is a 0-neigbhourhood in $X \otimes Y$ by (3) and it clearly maps into W under ψ. Thus ψ is continuous.

Note that 4 of the proposition says that projective tensor product topology is the strongest locally convex topology on $X \otimes Y$ for which the bilinear map $\theta : X \times Y \to X \otimes Y$ is continuous.

Note that of the above proposition says that projective tensor product topology is the strongest locally convex topology on $X \otimes Y$ for which determines its topology. Also, if X and Y are metrizable then so is $X \otimes Y$.

If X, Y and Z are locally convex spaces andn $\alpha : X \to Y$ is a continuous linear map then the composition

$$X \times Z \to Y \times Z \to Y \otimes_\pi Z$$

is a continuous bilinear map $X \times Z \to Y \times Z \to Y \otimes_\pi Z$ and, by Proposition 5.(4), it factors through a unique continuous linear map $\alpha \otimes id : X \otimes_\pi Z \to Y \otimes_\pi Z$. This shows that, for a fixed l.c.s. Z, $(.) \otimes_\pi Z$ is a functor from the category of locally convex spaces to itself. Similarly, the projective tensor product is also a functor in its second argument for each fixed l.c.s. appearing in its first argument.

3. Proposition

If $\alpha : X \to Y$ is a continuous linear open map, then so is $\alpha \otimes id : X \times Z \to Y \otimes Z$

Proof. To show that $\alpha \otimes id$ is open we must that each 0- neighbourhood in $X \otimes Z$ maps to a 0-neighbourhood in $Y \otimes Z$. However, this follows immediately from Proposition and the hypothesis that α is an open map.

The space $X \otimes_\pi Y$ is generally not complete. It is usually useful to complete it.

Definition

The completion of $X \otimes_\pi Y$ will be denoted $X \widehat{\otimes}_\pi Y$ and will be called the completed projective tensor product of X and Y.

Note that if $\alpha : X \to Y$ is a continuous linear map, the map $\alpha \otimes id : X \widehat{\otimes}_\pi \to Y \widehat{\otimes}_\pi Z$. Even under the hypothesis of Proposition 3.6 this map is not generally a surjection. However, we do have:

4. Proposition

If X, X and Z are Frechet spaces and $\alpha : X \to Y$ is a surjective continuous linear map, then $\alpha \otimes id : X \widehat{\otimes}_\pi \to Y \widehat{\otimes}_\pi Z$.

Proof. By the open mapping theorem, the map a is open. Then $\alpha \otimes id$ is open by Proposition 3. Since, the topologies of X, Y and Z countable bases at 0 the same in true of $X \otimes_\pi Z$ and $Y \otimes_\pi Z$. However, an open map between metrizable t.v.s.'s has the property that every Cauchy sequence in the range has a subsequence which is the image of a Cauchy sequence in the domain. Since every point in the completion $X \widehat{\otimes}_\pi Z$ is the limit of a Cauchy sequence in $Y \widehat{\otimes}_\pi Z$, the result follows.

Obviously, the analogues of Proposition 3.6 and 3.8 with the roles of the left and right arguments reversed are also true.

There are other hypothesis under which the conclusion of the above Proposition is true and we will return to this question when we have developed the tools to prove such results [7-10].

In the case where X and Y are Fechet spaces, elements of the completed projective tensor product $X \widehat{\otimes}_\pi Y$ may be represented in a particularly useful form:

5. Proposition

The dual space of $X \widehat{\otimes}_\pi Y$ is naturally isomorphic to B(X,Y).

Proof : If $\theta : X \times Y \to X \widehat{\otimes}_\pi Y$ is the bilinear map defined by $\theta(x, y) = x \otimes y$ then $f \to f o \theta$ is clearly a linear map of $X \widehat{\otimes}_\pi Y$ to $B\,(X, Y)$. It is an isomorphism.

We now proceed to the second important way of toppologizing the tensor product of two locally convex spaces.

Let X_σ^* denote X^* with its weak-* tology - that is, with the weak topology that X induces on X*. With a similar meaning for Y*, we note that $X \otimes Y$ is naturally embedded in $B\ (X_\sigma^*, Y_\sigma^*)$. In fact, if $u = \sum x_i \otimes y_i \in X \otimes Y$ the map

$$(f,g) \to \phi_u(f,g) = \sum f(x_i)g(y_i): X^* \times Y^* \to \mathbb{C}$$

is a separately continuous bilinear form on $X_\sigma^* \times Y_\sigma^*$. If we choose the set $\{x_i\}$ to be linearly independent, then it easy to see that $\phi_u \neq 0\ u \neq 0$. Thus, $u \to \phi_u$ is an embedding of $X \otimes Y$ into $B\ (X_\sigma^*, Y_\sigma^*)$.

Definition

The topology of bi-equicontinuous convergence on $B\ (X_\sigma^*, Y_\sigma^*)$ is the topology of uniform convergence on sets of the form $A \times B$ where A and B are equicontinuous subsets of X^* and Y^*, respectively [9-11]. We denote by $B_e(X_\sigma^*, Y_\sigma^*)$ the space $B\ (X_\sigma^*, Y_\sigma^*)$ with this topology. We denote by $X \otimes_e Y$ the space $X \otimes Y$ with the topology it inherits from its natural embedding in $B_e(X_\sigma^*, Y_\sigma^*)$. Finally, we denote by $X \widehat{\otimes}_e Y$ the completion of $X \otimes_e Y$.

Note that a family of seminorms determining the topology on $B_e(X_\sigma^*, Y_\sigma^*)$ consists of the seminorms of the form $p_{A,B,}$ where A and B are equicontinous subsets of X^* and Y^*, respectively and :

$$p_{A,B}(\phi) = \sup\{|\phi\ (f,g)|: (f,g) \in A \times B\}$$

A typical 0- neigbhourhood in this topology has the form

$$V_{A,B} = \{\phi \in B(X_\sigma^*, Y_\sigma^*) : |\phi(f,g)| < 1 \forall f \in A, g \in B\}$$

For this to make sense, we need to know that each $\phi \in B(X_\sigma^*, Y_\sigma^*)$ is bounded on each set of the form $A \times B$ with A and B equicontinuous. However, note that every equicontnuous subset of X^* is contained in one of the form V^0 for V a-neigbhourhood and sets of the form V^0 are compact in X_σ^* by the Banach-Alaoglu theorem, thus, we may assume that A and B are compact and convex. Since an element $\phi \in B(X_\sigma^*, Y_\sigma^*)$ is separately continuous, it is bounded on $\{x\} \times B$ for each $x \in A$ and on $a \times \{y\}$ for each $y \in B$. It follows from the Banach-Steinhaus Theorem for compact convex sets that ϕ is bounded on $A \times B$. Thus, the seminorm $p_{A,B}$ is defined on all of $B(X_\sigma^*, Y_\sigma^*)$. Clearly, the topology generated by the family of such seminorms males $B_e(X_\sigma^*, Y_\sigma^*)$ into an l.c.s.

If $\alpha : X \to Y$ is a continuous linear map between l.c.s.'s and Z is any l.c.s. then the adjoint map $\alpha^*: Y_\sigma^* \to X_\sigma^*$ is also continuous and maps equicontinuous sets to equicontinuous sets. It follows that $\alpha^* : Y_\sigma^* \to X_\sigma^*$ is also continuous and maps equicontinuous sets to equicontinuous sets. It follows that α^* induces a continuous linear map

$$(\alpha^* \times id)': B_e(X_\sigma^*, Z_\sigma^*) \to B_e(Y_\sigma^*, Z_\sigma^*)$$

where

$$(\alpha^* \times id)'\ \phi(f,g) = \ \phi\ (\alpha^*(f).g)$$

Restricted to the image of $X \otimes Z$ in $B_e(X_\sigma^*, Z_\sigma^*)$, this map is just $\alpha \otimes id : X \otimes Z \to Y \otimes Z$. Thus, it follows that $\alpha \otimes id$ is continuous linear map from $X \otimes_e Z\ to\ Y \otimes_e Z$ and extends to a continuous linear map $\alpha\ \otimes id : X \widehat{\otimes}_e Z \to Y \widehat{\otimes}_e Z$ between their completion. Thus, for a fixed l.c.s. Z, $(.)\otimes_e Z$ are functions from the category of locally convex spaces to itself.

Lemma

If X and Y are l.c.s.'s and $\alpha : X \to Y$ is a topological isomorphism onto its range. Then under $\alpha^* : Y^* \to X^*$ each equicontinous set in X^* is the image of an equicontinuous set in Y^*.

Proof. Since α is a topological isomorphism onto its image, we may identify X with its image and consider it a subspace of Y. Then α is the inclusion and α^* the restriction map from functional on Y to functional on X. Thus, we must show that each equicontinuous set in X^* is the restsriction of an equicontinuous set in Y^*. Since the class of equicontinuous sets is closed under passing to subsets, it is enough to show that each equicontinuous set A in X^* is contained in the restriction of an equicontinuous set B in Y^*. Furthermore, we may assume that $A = U^0$ for a convex, balance 0-neibhourhood $U \subset$ X since every equicontnuous set is contained in one of this form. However, by Lemma 2.10, for each such neibhourhood U there is a convex, balanced 0 neighbourhood $V \subset Y$ such that $V \cap Y$=U. The proof will be complete if we can establish that U^0 is the image of v^o under the restriction map. To this end. we let py be the Minkowaski functional of V. Since $V \cap X = U$, the restriction of $py\ to\ X$ is the Minkoski functional p_U of U. Now if $f \in X^*$, it is easy to see that $f \in U^0$ if and only if $|f| \leq p_U$ (Problem 3.5). Thus, if $f \in U^0$ if and only if $|f| \leq p_U$ (Problem 3.5.) Thus, if $f \in U^0$ then $|f| \leq pv$. Now the Hahn-Banach theorem implies that f has an extension g to Y which satisfies $|g| \leq pv$. This, in turn, implies that $g \in V^0$. Thus, U^o is the restriction to X of V^o and the proof is complete[11-14].

REFFERENCES

[1] H. Apiola: *Duality between spaces of p-summing operators and characterization of nuclearity.* Math. Ann. **219**, (1974), 53-64.

[2] Q. Bu, J. Diestel: *Observations about the projective tensor product of Banach space _p ⊗X,* $1 < p < \infty$. Quaestiones Mathematicae **24** (2001),519.

[3] W. Congxin, Q. Bu: *K¨othe dual of Banach spaces _p[E]* ($1 \leq p < \infty$) *and Grothendieck space.* Comment. Math. Univ. Carolinae **34**, (2) (1993), 265-273.

[4] N. De Grande-De Kimpe: *Generalized Sequence spaces.* Bull. Soc. Math. Belgique, **23**(1971), 123-166.

[5] M. Gupta, Q. Bu: *On Banach-valued sequence spaces _p[X].* J. Anal. **2** (1994), 103-113.

[6] D. W. Dean: *The equation* $L(E,X^{**}) = L(E,X)^{**}$ *and the principle of the local reflexivity.* Proc. Amer. Math. Soc., **40** (1973), 146-148.

[7] H. Jarchow: *Locally convex spaces.* B. G. Teubner Stuttgart(1981).

[8] G. K¨othe: *Topological Vector Spaces I and II.* Springer-Verlag, Berlin, Heidelberg, New York.

[9] M. A. Ould Sidaty: *Reflexivity and AK-property of certain vector sequence spaces.* Bull. Belg. Math. Soc., Simon Stevin 10 (4) (2003), 579-783.

[10] M. Florencio, P. J. Pa´ul: *Una representaci´on de ciertos ε-productos tensoriales*. Actas de las Jornadas Matematicas Hispano Lusas. Murcia (1985), 191-203.

[11] M. Florencio, P. J. Pa´ul: *La propiedad AK en ciertos espacios de suecsiones vectoriales.* Proc. Eleventh Spanish-Portuguese Conference on Mathematics. **1**, 197-203, Dep. Mat. Univ. Extremadura, **18**, (1987).

[12] M. Florencio, P. J. Pa´ul: *Barrelledness conditions on vector valued sequence spaces* Arch. Math. (Basel) Vol. **48** (1987), 153-164.

[13] M. Florencio, P. J. Pa´ul: *A note on λ-multiplier convergente series.* Casopis Pest. Mat.,**113** (1988), 421-428.

[14] A. Pietsch: *Nuclear locally convex spaces.* Springer-Verlag, Berlin, Heidelberg, New York (1972).