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Modified energy states of Harmonic Oscillator for a new Dispersion relation

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Abstract

In order to include the expected Planck-scale correction in the physical systems, we have put forward a novel modified dispersion relation (MDR).It has Lorentz invariance. A toy model like relativistic harmonic oscillator has been studied to get the necessary Planck-scale correction. It has been found that each level of harmonic oscillator acquires Planck-scale correction.

I. INTRODUCTION

The study of the probable scenario of different physical systems in the vicinity of Planck-scale is of huge interest since the strength of the gravitational interaction in that scale becomes comparable to the strength of the electromagnetic interaction. So, in the vicinity of Planck-scale, it necessities to take into account the effect of (quantum) gravity. But straightforward quantization of gravity and its direct insertion into the physical systems are still lying at the far-reaching stage. Therefore, the an indirect way of including the quantum gravity effect has been receiving much attention over the years through the use of novel ideas like generalized Heisenberg uncertainty principle and modified dispersion relation (MDR), as these two important ideas have been playing a remarkable role in providing necessary Planck-scale corrections into the physical systems in an indirect manner [1–9, 14]. This issue in the context of black body radiation was fond to be addressed in the articles [16–19]. To make statistical mechanics compatible to incorporate quantum gravity effect, formal development of it along with few applications in the thermodynamical systems have been explored in the articles [20–24]. Some experimental result also have been found to be reported to explain with the use of this type of generalization (modification)[25, 26].

In the articles [16–21, 23, 24, 27–31] the concept of GUP have been exercised extensively to incorporate Planck-scale effect into different physical systems and the concept of MDR has been extended for the same purpose in the articles [32–38]. Although these two novel concepts (GUP and MDR) have been exercised in independent manner to serve essentially the same purpose in different physical systems, it is not difficult to understand conceptually that MDR and GUP are complementary to each other, however, it is fair to admit that there is no established direct one to one correspondence between these two in general. It depends on the choice of generalization of the uncertainty relation or on the choice of modification of the dispersion relation. We should mention at this stage that an attempt to establish the conceptual connection between GUP and MDR is carried out recently in the article [40]. There is indeed a special instance where a precise MDR has been proposed for a specific GUP [42].

Although these two decent, as well as potentially effective ideas, got strong initial support from string theory [2, 4, 43, 44, 46–48], these two ideas were also supported immensely by the other quantum gravity candidates like loop quantum gravity [8, 45] and space-time non-commutativity [49–52]. In these two modifications (though in principle should be considered to be merged into one) Lorentz symmetry was needed to be ignored and that necessarily led to open up a new idea namely deformed special relativity (DSR) [53]. So a natural question may arise whether this modification can be made maintaining Lorentz symmetry. It is true that there are some experimental signature which was explained inviting the Lorentz violation in an essential way [39], but it would be nicer indeed if it could be explained maintaining the Lorentz symmetry since violation of Lorentz symmetry invites unwanted problems to a great extent because this symmetry is deeply rooted both in field theory as well as in the general theory of relativity. The articles [10–12] also shows a possibility of framing MDR in a Lorentz symmetric manner where deformation of d^0 Alembertian has been exercised. In the article [10] a scenario of significant interest is found where d'Alembertian is deformed into a function of itself. The structure is very valuable and

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suitable for construction of Lorentz invariant MDR with which we will deal here. It indeed has intrinsic interest since it has been proposed on the basis of studies of the Asymptotic-Safety approach [10] and of the approach based on Causal Sets [13]. We use this here because it includes quantum gravity perspective in a significant manner as has been emphasized in the article [10].

In this article, we, therefore, introduce a generalized modified dispersion relation in such a way that it can be used in Lorentz invariant manner as well as without maintaining that invariance. In this context, we should mention that in the articles [54, 55], relativistic quantum mechanical systems without quantum gravity correction have been studied and at the same time the important article [56], is of the worth mentionable where the possibility of incorporating invariant quantum gravity effect at the vicinity of Planck-scale has been discussed in detail.

So it would be beneficial to design a generalized MDR in such a fashion so that it can meet both the purpose: Lorentz invariance as well as Lorentz non-invariance extension of any physical system towards incorporating the Planck-scale effect of any physical system. With the MDR proposed here, we have studied the toy model like a relativistic one-dimensional harmonic oscillator to include the Planck-scale correction into their energy spectrum. Recently, this system has been studied in [57, 58] for the same purpose with the use of GUP.

II. FORMULATION OF NEW DISPERSION RELATION TO GET THE PLANCK-SCALE CORRECTIOM

The generalized MDR which we are going to formulate in order to incorporate quantum gravity effect can be casted in Lorentz invariant as well as Lorentz non-invariant manner. So if the extension with this MDR is carried in a Lorentz symmetric manner one need not be worried about the search of precise DSR corresponding to this MDR. The explicit expression of this generalized MDR is

$$\mathbf{P}^{\mu}\mathbf{P}_{\mu} = m_0^2 + \sum_k f_k (p^2 - E^2)^k. \qquad (1)$$

Here $f_k = \frac{\alpha_k}{(m_P)^k} g(\frac{p}{E})$. In general, any Lorentz non-covariant structure of the function $g(\frac{p}{E})$ breaks the Lorentz symmetry. However any p independent or Lorentz covariant structure of $g(\frac{p}{E})$ preserves Lorentz invariance. So this new generalized MDR can be used in any physical system to incorporate Planck-scale correction in significant manner.

We have chosen the simplest possible Lorentz covariant structure of the function $g(\frac{p}{E})$, as $g(\frac{p}{E}) = 1$, which makes $f_k = \frac{\alpha_k}{(m_P)^k}$. So the MDR with which we will start our investigation is

$$\mathbf{P}^{\mu}\mathbf{P}_{\mu} = m_0^2 + \sum_k \frac{\alpha_k}{(m_P)^k} (\mathbf{P}^{\mu}\mathbf{P}_{\mu})^k. \qquad (2)$$

Here m_0 is the rest mass of the particle considered for study, k is an integer that runs from $k = 2$ to any desired order N, $P = (E, p\sim)$ and $p = |p\sim|$. The parameter α_k represents arbitrary N-2 number of free parameters which can be fixed comparing the result obtained using this MDR with the experimental result or with the results obtained earlier through other reliable concept like GUP. The Planck-mass is characterized by the symbol m_p which is given by $m_P = \sqrt{\frac{\hbar c}{G}}$. Note that, reparametrization invariance in the action of a system can be maintained with this MDR (2) having manifestly lorentz covariance, since under $p \to -p$ this MDR remains unchanged. Of course, we can define (2) in terms of Planck-length l_P in the following way

$$\mathbf{P}^{\mu}\mathbf{P}_{\mu} = m_0^2 + \sum_k \eta_k (l_P)^k (\mathbf{P}^{\mu}\mathbf{P}_{\mu})^k, \qquad (3)$$

where η_k represents $N - 2$ number of parameters certainly different from α_k and the Planck-length $l_P = \sqrt{\frac{\hbar}{Gc^3}}$. In general, there are infinite number of parameters and numerical computation is possible with an arbitrary large numbers of such parameters, although analytical calculation will not always be possible with a desired numbers of terms. In practice however we need not keep all these parameter to get the desired accuracy.

The symbol c stands for velocity of light in vacuum in both the cases. We will choose natural unit ~ = 1 and $c = 1$. To carry out our analytical investigations on relativistic harmonic oscillator with this generalized MDR we will keep ourselves limited to $k = 2$. In due course, we will find that with the terms available for the the choice $k = 2$, the analytical computation towards Planck-scale correction for the said systems is tractable and it has good agreement with the result of the system studied earlier using the concept of GUP to incorporate the Planck-scale effect [9]. The MDR with $k = 2$ reads

$$P^2 = E^2 + m_0^2 + \frac{\alpha_2}{(m_P)^2}(E^2 - P^2)^2. \qquad (4)$$

A careful look reveals that this MDR resembles the deformation of the d^0 Alembertian () = $\partial_\mu \partial^\mu$ as a function of itself as has been addressed in [10–12]. This was adopted there in connection to the spectral dimension which in turn shows short distance behavior. To be more precise: to include the dynamics of a physical system for any energy scale the deformed d^0 Alembertian would be some desired function of the usual d^0 Alembertian ($f(\Box)$) having some constants that has to be fixed with the experimental result. So to get Plank-scale effect it is to be considered that the dynamics below the Plank scale will be governed with the undeformed d^0 Alembertian and the dynamics in the vicinity will be governed by the usual d^0 Alembertian along with deformation part of the d^0 Alembertian in a judicious manner to get necessary agreement with the experimental result (if or when available). Since both the has bears the Lorentz symmetry the frame independence of the physical result will be protected. Of course velocity of light will remain as an invariant quantity.

The hitherto available literature related MDR show that the MDR's contain the expression of E as a function of P. This new MDR is not an exception to that, but it is true that the nature of the function is a little generalized. The modification is made here is only the enforcement of a physical symmetry which is none other than the celebrated Lorentz invariance which was not there in the MDR designed earlier. We have kept this point of view that in the vicinity of the Planck-scale violation of Lorentz invariance may occur but it can not be a basic criterion. On the other hands it would be admitted from all corners that Lorentz the covariant structure is advantageous over Lorentz noncovariance in many respect because of the basic foundation of quantum field theory and the general theory of relativity is deeply rooted in the Poincare symmetry. The article [15] is an excellent example in this direction, therefore, this new MDR will certainly add new light in the formal development of MDR related theories. The novelty of this MDR is the welcome entry of the Lorentz symmetry and its capability to render Planck-scale correction in a frame an independent manner which is lacking in the construction of MDR in the available literature dealing with Planck-scale correction with MDR.

Taking the square root of the above expression we get

$$E = \sqrt{p^2 + m_0^2 + \frac{\alpha_2}{(m_P)^2}(p^2 - E^2)^2} \quad (5)$$

Equation (5) on binomial expansion results

$$\begin{aligned} E &= \frac{p^2}{2m_0} + m_0 + \frac{\alpha_2 E_n^4}{2m_0(m_P)^2} - p^2\left(\frac{\alpha_2 E_n^4}{4m_0^3(m_P)^2} + \frac{\alpha_2 E_n^2}{m_0(m_P)^2}\right) \\ &+ p^4\left(\frac{\alpha_2}{2m_0(m_P)^2} + \frac{\alpha_2 E_n^2}{2m_0^3(m_P)^2} - \frac{1}{8(m_0)^3} + \frac{3\alpha_2 E_n^4}{16m_0^5(m_P)^2}\right) \\ &+ p^6\left(-\frac{\alpha_2}{4m_0^3(m_P)^2} + \frac{1}{16(m_0)^5} - \frac{3\alpha_2 E_n^2}{8m_0^5(m_P)^2} - \frac{5\alpha_2 E_n^4}{32m_0^7(m_P)^2}\right). \end{aligned} \quad (6)$$

In the above expansion we have neglected terms containing higher order in α_2 and retained the terms up to sixth order in p. Thus our modified Hamiltonian with this settings reads

$$\begin{aligned} H &= \frac{p^2}{2m_0} + V + m_0 + \frac{\alpha E_n^4}{2m_0(m_P)^2} - p^2\left(\frac{\alpha E_n^4}{4m_0^3(m_P)^2} + \frac{\alpha E_n^2}{m_0(m_P)^2}\right) \\ &+ p^4\left(\frac{\alpha}{2m_0(m_P)^2} + \frac{\alpha E_n^2}{2m_0^3(m_P)^2} - \frac{1}{8(m_0)^3} + \frac{3\alpha E_n^4}{16m_0^5(m_P)^2}\right) \\ &+ p^6\left(-\frac{\alpha}{4m_0^3(m_P)^2} + \frac{1}{16(m_0)^5} - \frac{3\alpha E_n^2}{8m_0^5(m_P)^2} - \frac{5\alpha E_n^4}{32m_0^7(m_P)^2}\right). \end{aligned} \quad (7)$$

In the above expression we have replaced α_2 by α since there is no other α's in the body of the article. Here V stands for the potential energy inserted by hand. To avoid confusion we would like to mention that $[q, p] = i\hbar$ and for modified position and momentum pair (q_m, p_m) the canonical Poission's bracket will take a modified form and that will lead to a modified uncertainty relation having a generalized form $[q_m, p_m] = i\hbar f(p,E)$. The precise form of the function $f(p,E)$ will certainly depend on the nature of the choice of the MDR. Now using this Hamiltonian we will proceed to calculate modified eigenvalues for the relativistic harmonic oscillator. The modified eigenvalues are in general given by

$$\begin{aligned} {E_n}^{(m)} &= E_n + m_0 + \frac{\alpha E_n^4}{2m_0(m_P)^2} - \langle p^2 \rangle \left(\frac{\alpha E_n^4}{4m_0^3(m_P)^2} + \frac{\alpha E_n^2}{m_0(m_P)^2}\right) \\ &+ \langle p^4 \rangle \left(\frac{\alpha}{2m_0(m_P)^2} + \frac{\alpha E_n^2}{2m_0^3(m_P)^2} - \frac{1}{8(m_0)^3} + \frac{3\alpha E_n^4}{16m_0^5(m_P)^2}\right) \\ &+ \langle p^6 \rangle \left(-\frac{\alpha}{4m_0^3(m_P)^2} + \frac{1}{16(m_0)^5} - \frac{3\alpha E_n^2}{8m_0^5(m_P)^2} - \frac{5\alpha E_n^4}{32m_0^7(m_P)^2}\right), \end{aligned} \quad (8)$$

where $E_n^{(m)}$ and E_n are modified and unmodified eigenvalues respectively. This generalized expression shows that we have to evaluate the expectation values up to sixth power of momentum. To be precise we need the expressions of $\langle p^2 \rangle$, $\langle p^4 \rangle$, and $\langle p^6 \rangle$ of the system that would be considered under investigation.

III. RELATIVISTIC HARMONIC OSCILLATOR WITH MDR USING RAISING AND LOWERING REPRESENTATION OF MOMENTUM OPERATOR

Let us first proceed to calculate the energy eigenvalues using raising and lowering representation of the momentum. It is known that the raising and lowering operators are respectively given by

$$a = \frac{1}{\sqrt{2m_0\omega\hbar}}(m_0\omega x + ip) \quad (9)$$

$$a^\dagger = \frac{1}{\sqrt{2m_0\omega\hbar}}(m_0\omega x - ip). \quad (10)$$

For the sake of convenience we will not set ~ = 1 but c = 1 will be maintained from the starting point. However the final expression will be presented with ~ = 1. If a and a^+ are operated separately on the n^{th} eigen state $|n>$ we get

$$a|n> = \sqrt{n}|n-1>, \quad (11)$$

$$a^\dagger|n> = \sqrt{n+1}|n+1>. \quad (12)$$

The momentum operator in terms of raising and lowering operator can be expressed as

$$p = i\sqrt{\frac{m_0\omega\hbar}{2}}(a^\dagger - a) \quad (13)$$

Consecutive operation of p on the n^{th} eigen state for two, four and six times results the following

$$p^2|n> = -\frac{m_0\omega\hbar}{2}[\sqrt{n+1}\sqrt{n+2}|n+2> -(2n+1)|n> +\sqrt{n}\sqrt{n-1}|n-2>] \quad (14)$$

$$\begin{aligned} p^4|n> &= (\frac{m_0\omega\hbar}{2})^2[\sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\sqrt{n+4}|n+4> \\ &- \sqrt{n+1}\sqrt{n+2}(4n+6)|n+2> \\ &+ (6n^2+6n+3)|n> -\sqrt{n}\sqrt{n-1}(4n-2)|n-2> \\ &+ \sqrt{n}\sqrt{n-1}\sqrt{n-2}\sqrt{n-3}|n-4>], \end{aligned} \quad (15)$$

$$\begin{aligned} p^6|n> &= -(\frac{m_0\omega\hbar}{2})^3[\sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\sqrt{n+4}\sqrt{n+5}\sqrt{n+6}|n+6> \\ &- \sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\sqrt{n+4}(6n+15)|n+4> \\ &+ \sqrt{n+1}\sqrt{n+2}(15n^2+45n+45)|n+2> \\ &- (20n^3+30n^2+40n+15)|n> \\ &+ \sqrt{n}\sqrt{n-1}(15n^2-15n+15)|n-2> \\ &- \sqrt{n}\sqrt{n-1}\sqrt{n-2}\sqrt{n-3}(6n-9)|n-4> \\ &+ \sqrt{n}\sqrt{n-1}\sqrt{n-2}\sqrt{n-3}\sqrt{n-4}\sqrt{n-5}|n-6>]. \end{aligned} \quad (16)$$

If we now take the inner product of the above expression with $|n>$ we will get the required expectation values of p^2, p^4 and p^6. The precise expression of the expectation values are

$$<p^2> = \frac{m_0\omega\hbar}{2}(2n+1), \quad (17)$$

$$<p^4> = (\frac{m_0\omega\hbar}{2})^2(6n^2+6n+3), \quad (18)$$

$$<p^6> = (\frac{m_0\omega\hbar}{2})^3(20n^3+30n^2+40n+15) \quad (19)$$

which leads us to get the modified eigenvalues with this Lorentz symmetric modified enegy values are

$$\begin{aligned} {E_n}^{(mho)} &= E_n + m_0 + \frac{\alpha E_n^4}{2m_0(m_P)^2} - \frac{m_0\omega\hbar}{2}(2n+1)(\frac{\alpha E_n^4}{4m_0^3(m_P)^2} + \frac{\alpha E_n^2}{m_0(m_P)^2}) \\ &+ (\frac{m_0\omega\hbar}{2})^2(6n^2+6n+3)(\frac{\alpha}{2m_0(m_P)^2} + \frac{\alpha E_n^2}{2m_0^3(m_P)^2} - \frac{1}{8(m_0)^3} + \frac{3\alpha E_n^4}{16m_0^5(m_P)^2}) \\ &+ (\frac{m_0\omega\hbar}{2})^3 \times (20n^3+30n^2+40n+15) \\ &(-\frac{\alpha}{4m_0^3(m_P)^2} + \frac{1}{16(m_0)^5} - \frac{3\alpha E_n^2}{8m_0^5(m_P)^2} - \frac{5\alpha E_n^4}{32m_0^7(m_P)^2}). \end{aligned} \quad (20)$$

Equation (20) reveals that each levels of harmonic oscillator acquires Planck-scale correction which was also found in the articles [57, 58] where concept of GUP was employed to incorporate quantum gravity effect. This result is amenable to compare with the correction obtained in [9], using the concept of GUP for incorporation of quantum gravity correction.

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