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Coupled Best Proximity Point Theorem in Metric Spaces

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Abstract:- The purpose of this article is to generalized the result of W. Sintunavarat and P. Kumam [4]. We also give an example in support of our theorem for which result of W. Sintunavarat and P. Kumam [4] is not true. Moreover we establish the existence and convergence theorems of coupled best proximity points in metric spaces, we apply this results in a uniformly convex Banach space.

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Keywords- Coupled fixed point, Coupled Common Fixed Point, Coupled best proximity point, Mixed monotone.

1. Introduction and Preliminaries

The metric fixed point theory is very important and useful in Mathematics. It can be applied in various areas to find the solution, for instant, in computer science, optimization, approximation theory, image processing as well as in economical problems. The first result of fixed point theorem is given by Banach S. [4] by the general setting of complete metric space using which is known as Banach Contraction Principle. There are many researchers generalized this contraction principle in different directions

In 1969, one of the most beautiful generalization of Banach contraction principle [1] is presented by Fan [2] which is known as best approximation theorem.

Theorem-1 If A is a nonempty convex subset of a Hausdorff locally convex topological vector space B and $S: A \to B$ is continuous mapping, then there exists an element $x \in A$ such that $d(x, Sx) = d(Sx, A)$.

The concept of coupled best proximity point theorem is introduced by W. Sintunavarat and P. Kumam [4] and proved coupled best proximity theorem for cyclic contraction. It should be clear that we can find a best proximity point in place of fixed point, if the fixed point does not exist. This best proximity point is much closer to the fixed point. If this distance is equal to zero then the best proximity point is called fixed point. Here one of the two things is important for best proximity point either distance must be equal to zero or very near to zero. If this condition does not exist then the point is not a best proximity point. In this condition we move to find another function which provides the distance must be closed to zero. So our purpose of this article is to generalized the result of [4] also we give an example in support of our main theorem.

Now we recall some basic definitions and examples that are related to the main results of this article.

Throughout this article we denote by N the set of all positive integers and by R the set of all real numbers. For nonempty subsets A and B of a metric space (X, d), we let

$$d(A, B) = inf \ \{d(x, y) : x \in A \ and \ y \in B\}$$

stands for the distance between A and B.

A Banach spaces X is said to be

i. *strictly convex* if the following implication holds for all $x, y \in X$:

$$\| x \| = \| y \| = 1 \ and \ x \neq y \Rightarrow \left\| \frac{x+y}{2} \right\| < 1.$$

ii. *uniformly convex* if for each ϵ with $0 < \epsilon \leq 2$, there exists $\delta > 0$ such that the following implication holds for all $x, y \in X$:

$$\| x \| \leq 1, \| y \| \leq 1 \ and \ \|x - y\| \geq \epsilon \Rightarrow \left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

It is easily to see that a uniformly convex Banach space X is strictly but the converges is not true.

Definition-2 [7] Let A and B be nonempty subsets of a metric space (X, d). The ordered pair (A, B) satisfies the property UC if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$, then $d(x_n, z_n) \to 0$.

Example-3 Let A and B be nonempty subsets of a metric space (X, d). The following are examples of a pair of nonempty subsets (A, B) satisfying the property UC.

i. Every pair of nonempty subsets A,B of a metric space (X,d) such that d(A,B) = 0.

ii. Every pair of nonempty subsets A,B of a uniformly convex Banach space X such that A is convex.

iii. Every pair of nonempty subsets A,B of a strictly convex Banach space which A is convex and relatively compact and the closure of B is weakly compact.

Definition- 4[5] Let A and B be nonempty subsets of a metric space (X,d). The ordered pair (A,B) satisfies the property UC^* if (A,B) has property UC and the following condition holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B satisfying:

i. $d(z_n, y_n) \to d(A, B)$

ii. For every $\epsilon > 0$ there exists $N \in \mathcal{N}$ such that

$$d(x_m, y_n) \leq d(A, B) + \epsilon$$

for all $m > n \geq N$.

Then for every $\epsilon > 0$ there exists $N_1 \in \mathcal{N}$ such that

$$d(x_m, z_n) \leq d(A, B) + \epsilon$$

for all $m > n \geq N_1$.

Example-5[5] Let A and B be nonempty subsets of a metric space (X,d).

The following are examples of a pair of nonempty subsets (A,B) satisfying the property UC^*.

i. Every pair of nonempty subsets A,B of a metric space (X,d) such that $d(A, B) = 0$.

ii. Every pair of nonempty closed subsets A,B of a uniformly convex Banach space X such that A is convex.

Definition-6 Let A and B be nonempty subsets of a metric space (X,d) and $S: A \to B$ be a mapping. A point $x \in A$ is said to be a best proximity point of S if it satisfies the condition that

$$d(x, Sx) = d(A, B).$$

It can be observed that a best proximity point reduces to a fixed point if the underlying mapping is a self mapping.

Definition- 7 Let A be a nonempty subset of a metric space X and $F: A\times A \to A$. A point $(x,y)\in A\times A$ is called a coupled fixed point of F if

$$x = F(x,y), \quad y = F(y,x).$$

2. Coupled best proximity point theorems

In this section we study the existence and convergence of coupled best proximity points for cyclic contraction pair.

Definition-8 Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be ICS if T is injective, continuous and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.

In this paper we give some coupled best proximity point theorems for mapping having the mixed monotone property in partially ordered metric space depended on another function, called T-cyclic contraction which is generalization of the main results of W. Sintunavarat and P. Kumam [4].

Definition-9 Let A and B be nonempty subsets of a metric space X and $F: A\times A \to B$.

An ordered coupled $(x,y)\in A\times A$ is called a coupled best proximity point of F if,

$$d(x,F(x,y)) = d(y,F(y,x)) = d(A,B).$$

It is easy to see that if $A = B$ in Definition-9, then a coupled best proximity point reduces to a coupled fixed point.

Next,W. Sintunavarat and P. Kumam [4] introduce the notion of a cyclic contraction for two mappings which as follows.

Definition-10 Let A and B be nonempty subsets of a metric space X, $F: A\times A \to B$ and $G: B\times B \to A$. The ordered pair (F,G) is said to be a cyclic contraction if there exists a non-negative number $\alpha < 1$ such that

$$d(F(x,y),G(u,v)) \le \frac{\alpha}{2}\,[d(x,u) + d(y,v)] + (1-\alpha)d(A,B)$$

for all $(x,y)\in A\times A$ and $(u,v)\in B\times B$.

Now we introduced the following notion of T-cyclic contraction for two mappings which is generalization of [4] as follows.

Definition- 11 Let T be an ICS mapping such that $T: X \to X$ and let A and B be nonempty subsets of a metric space X, $F: A\times A \to B$ and $G: B\times B \to A$. The ordered pair (F,G) is said to be a T-cyclic contraction if there exists a non-negative number $\alpha < 1$ such that

$$d(TF(x,y),TG(u,v)) \le \frac{\alpha}{2}\,[d(Tx,Tu) + d(Ty,Tv)] + (1-\alpha)T(d(A,B))$$

for all $(x,y)\in A\times A$ and $(u,v)\in B\times B$.

Note that if (F,G) is a T-cyclic contraction, then (G,F) is also a T-cyclic contraction. Also if we take T be an identity mapping in Definition-11 then we get Definition-10.

Following example show that Definition-11 is generalization of Definition-10.

Example-12 Let $X = R$ with the usual metric $d(x,y) = |x-y|$ and $Tx = x+1$ also $A = \left[\frac{3}{2},\frac{5}{2}\right]$ and $B = \left[-\frac{5}{2},-\frac{3}{2}\right]$. It easy to see that $d(A,B) = 3$. Define $F: A\times A \to B$ and $G: B\times B \to A$ by

$$F(x,y) = \frac{x-y-7}{4} \quad \text{and} \quad G(x,y) = \frac{x-y+1}{4}.$$

For arbitrary $(x,y)\in A\times A$, $(u,v)\in B\times B$ and fixed $\alpha = \frac{1}{2}$, we get

$$\begin{aligned} d(TF(x,y),TG(u,v)) &= \left|\frac{x-y-7+4}{4} - \frac{u-v+1+4}{4}\right| \\ &\le \frac{|x-u|+|y-v|}{4} + 2 \\ &= \frac{\alpha}{2}[d(Tx,Tu) + d(Ty,Tv)] + (1-\alpha)T(d(A,B)). \end{aligned}$$

This implies that (F,G) is a T- cyclic contraction with $\alpha = \frac{1}{2}$

The following lemma plays an important role in our main results.

Lemma- 13 Let $T: X \to X$ be an ICS mapping also A and B be nonempty subsets of a metric space X, $F: A\times A \to B$ and $G: B\times B \to A$ and (F,G) be a T-cyclic contraction. If $(x_0,y_0)\in A\times A$ and we define

$$x_{n+1} = F(x_n, y_n), \quad x_{n+2} = G(x_{n+1}, y_{n+1})$$
$$y_{n+1} = F(y_n, x_n), \; y_{n+2} = G(y_{n+1}, x_{n+1})$$

for all $n \in N \cup \{0\}$, then $d(x_n, x_{n+1}) \to d(A,B), d(x_{n+1}, x_{n+2}) \to d(A,B), d(y_n, y_{n+1}) \to d(A,B)$ and $d(y_{n+1}, y_{n+2}) \to d(A,B)$.

Proof: For each $n \in N$, we have

$$d(Tx_n, Tx_{n+1}) = d\big(TF(x_n, y_n), TG(x_{n-1}, y_{n-1})\big)$$
$$\leq \frac{\alpha}{2}\,[d(Tx_n, Tx_{n-1}) + d(Ty_n, Ty_{n-1})] + (1-\alpha)T\big(d(A,B)\big)$$

Similarly we have

$$d(Ty_n, Ty_{n+1}) = d\big(TF(y_n, x_n), TG(y_{n-1}, x_{n-1})\big)$$
$$\leq \frac{\alpha}{2}\,[d(Ty_n, Ty_{n-1}) + d(Tx_n, Tx_{n-1})] + (1-\alpha)T\big(d(A,B)\big)$$

Therefore, by letting

$$d_n = d(Tx_n, Tx_{n+1}) + d(Ty_n, Ty_{n+1})$$

by adding above inequality we have

$$d_n \leq \alpha\, d_{n-1} + 2(1-\alpha)T\big(d(A,B)\big)$$

Similarly we can show that

$$d_{n-1} \leq \alpha\, d_{n-2} + 2(1-\alpha)T\big(d(A,B)\big)$$

Consequently we have

$$d_1 \leq \alpha\, d_0 + 2(1-\alpha)T\big(d(A,B)\big)$$

If $d_0 = 0$ then (x_0, y_0) is a coupled best proximity point of F and G. Now let $d_0 > 0$ for each $n \geq m$ we have

$$d(Tx_n, Tx_m) \leq d(Tx_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_{n-2}) + \ldots\ldots\ldots + d(Tx_{m+1}, Tx_m)$$
$$d(Ty_n, Ty_m) \leq d(Ty_n, Ty_{n-1}) + d(Ty_{n-1}, Ty_{n-2}) + \ldots\ldots\ldots + d(Ty_{m+1}, Ty_m)$$

which gives

$$d(Tx_n, Tx_m) + d(Ty_n, Ty_m) \leq d_{n-1} + d_{n-2} + d_{n-3} \ldots\ldots + d_m$$
$$d_n \leq \alpha^n\, d_0 + 2^n(1-\alpha^n)\,T\big(d(A,B)\big)$$

Taking $n \to \infty$ we have

$$d(Tx_n, Tx_{n+1}) + d(Ty_n, Ty_{n+1}) \to T\big(d(A,B)\big)$$

implies that

$$d(Tx_n, Tx_{n+1}) \to T\big(d(A,B)\big)$$
$$d(Ty_n, Ty_{n+1}) \to T\big(d(A,B)\big)$$

for all $n \in N$.

By similar argument, we also have

$$d(Tx_{n+1}, Tx_{n+2}) \to T\big(d(A,B)\big), \text{ and } d(Ty_{n+1}, Ty_{n+2}) \to T\big(d(A,B)\big).$$

Since T is injective mapping so we have

$$d(x_n, x_{n+1}) \to d(A,B) \text{ and } d(y_n, y_{n+1}) \to d(A,B)$$

for all $n \in N$.

By similar argument, we also have

$$d(x_{n+1}, x_{n+2}) \to d(A,B), \text{ and } d(y_{n+1}, y_{n+2}) \to d(A,B).$$

Lemma – 14 Let $T: X \to X$ be an ICS mapping also let A and B be nonempty subsets of a metric space X such that (A,B) and (B,A) have a property UC, $F: A \times A \to B$ and $G: B \times B \to A$ and let the ordered pair (F,G) is a T- cyclic contraction. If $(x_0, y_0) \in A \times A$ and define

$$x_{n+1} = F(x_n, y_n), \; x_{n+2} = G(x_{n+1}, y_{n+1}) \text{ and } y_{n+1} = F(y_n, x_n), \; y_{n+2} = G(y_{n+1}, x_{n+1})$$

for all $n \in N \cup \{0\}$, then for $\epsilon > 0$, there exists a positive integer N_0 such that for all $m > n \geq N_0$

$$\frac{\alpha}{2}[d(Tx_m, Tx_{n+1}) + d(Ty_m, Ty_{n+1})] < T(d(A,B)) + \epsilon. \quad (2.1)$$

Proof : By Lemma-13, we have

$$d(Tx_n, Tx_{n+1}) \to T(d(A,B)), \quad d(Tx_{n+1}, Tx_{n+2}) \to T(d(A,B)),$$

$$d(Ty_n, Ty_{n+1}) \to T(d(A,B)), \quad d(Ty_{n+1}, Ty_{n+2}) \to T(d(A,B)).$$

Since (A,B) has a property UC, we get

$$d(Tx_n, Tx_{n+2}) \to 0.$$

A similar argument shows that

$$d(Ty_n, Ty_{n+2}) \to 0.$$

As (B,A) has a property UC, we also have

$$d(Tx_{n+1}, Tx_{n+3}) \to 0, \quad d(Ty_{n+1}, Ty_{n+3}) \to 0.$$

Suppose that (2.1) does not hold. Then there exists $\epsilon' > 0$ such that for all $k \in N$,there is $m_k > n_k \geq k$ satisfying

$$\frac{\alpha}{2}\left[d(Tx_{m_k}, Tx_{n_k+1}) + d(Ty_{m_k}, Ty_{n_k+1})\right] \geq T(d(A,B)) + \epsilon'.$$

Further, corresponding to n_k , we can choose m_k in such a way that it is the smallest integer with $m_k > n_k$ and satisfying above relation.

Then

$$\frac{\alpha}{2}\left[d(Tx_{m_k-2}, Tx_{n_k+1}) + d(Ty_{m_k-2}, Ty_{n_k+1})\right] < T(d(A,B)) + \epsilon'.$$

Therefore, we get

$$\begin{aligned} T(d(A,B)) + \epsilon' &\leq \frac{\alpha}{2}\left[d(Tx_{m_k}, Tx_{n_k+1}) + d(Ty_{m_k}, Ty_{n_k+1})\right] \\ &\leq \frac{\alpha}{2}\left[d(Tx_{m_k}, Tx_{m_k-2}) + d(Tx_{m_k-2}, Tx_{n_k+1})\right] + \frac{\alpha}{2}\left[d(Ty_{m_k}, Ty_{m_k-2}) + d(Ty_{m_k-2}, Ty_{n_k+1})\right] \\ &< \frac{\alpha}{2}\left[d(Tx_{m_k}, Tx_{m_k-2}) + d(Ty_{m_k}, Ty_{m_k-2})\right] + T(d(A,B)) + \epsilon'. \end{aligned}$$

Letting $k \to \infty$, we obtain to see that

$$\frac{\alpha}{2}\left[d(Tx_{m_k}, Tx_{n_k+1}) + d(Ty_{m_k}, Ty_{n_k+1})\right] \to T(d(A,B)) + \epsilon'.$$

By using the triangle inequality, we get

$$\begin{aligned} \frac{\alpha}{2}\left[d(Tx_{m_k}, Tx_{n_k+1}) + d(Ty_{m_k}, Ty_{n_k+1})\right] &\leq \frac{\alpha}{2}\left[d(Tx_{m_k}, Tx_{m_k+2}) + d(Tx_{m_k+2}, Tx_{n_k+3}) + d(Tx_{n_k+3}, Tx_{n_k+1})\right] \\ &\quad + \frac{\alpha}{2}\left[d(Ty_{m_k}, Ty_{m_k+2}) + d(Ty_{m_k+2}, Ty_{n_k+3}) + d(Ty_{n_k+3}, Ty_{n_k+1})\right] \\ &= \frac{\alpha}{2}\left[\begin{matrix} d(Tx_{m_k}, Tx_{m_k+2}) + d\left(TG(x_{m_k+1}, y_{m_k+1}), TF(x_{n_k+2}, y_{n_k+2})\right) \\ + d(Tx_{n_k+3}, Tx_{n_k+1}) \end{matrix}\right] \\ &\quad + \frac{\alpha}{2}\left[\begin{matrix} d(Ty_{m_k}, Ty_{m_k+2}) + d\left(TG(y_{m_k+1}, x_{m_k+1}), TF(y_{n_k+2}, x_{n_k+2})\right) \\ + d(Ty_{n_k+3}, Ty_{n_k+1}) \end{matrix}\right] \\ &\leq \frac{\alpha}{2}[d(Tx_{m_k}, Tx_{m_k+2}) + \frac{\alpha}{2}\left[\begin{matrix} d(Tx_{m_k+1}, Tx_{n_k+2}) + d(Ty_{m_k+1}, Ty_{n_k+2}) \\ + (1-\alpha)T(d(A,B)) \end{matrix}\right] \\ &\quad + d(Tx_{n_k+3}, Tx_{n_k+1}) \\ &\quad + \frac{\alpha}{2}[d(Ty_{m_k}, Ty_{m_k+2}) + \frac{\alpha}{2}\left[\begin{matrix} d(Ty_{m_k+1}, Ty_{n_k+2}) + d(Tx_{m_k+1}, Tx_{n_k+2}) \\ + (1-\alpha)T(d(A,B)) \end{matrix}\right] \\ &\quad + d(Ty_{n_k+3}, Ty_{n_k+1}) \\ &\leq \alpha \left[\begin{matrix} d(Tx_{m_k}, Tx_{m_k+2}) + d(Tx_{n_k+3}, Tx_{n_k+1}) \\ + d(Ty_{m_k}, Ty_{m_k+2}) + d(Ty_{n_k+3}, Ty_{n_k+1}) \end{matrix}\right] \\ &\quad + \alpha^2\left[d(Tx_{m_k+1}, Tx_{n_k+2}) + d(Ty_{m_k+1}, Ty_{n_k+2})\right] + (1-\alpha^2)T(d(A,B)). \end{aligned}$$

Taking $k \to \infty$, we get

$$T(d(A,B)) + \epsilon' \leq \alpha^2 [T(d(A,B)) + \epsilon'] + (1 - \alpha^2)T(d(A,B)) = T(d(A,B)) + \alpha^2 \epsilon'$$

Since T is injective mapping so we have

$d(A,B) + \epsilon' \leq \alpha^2 [d(A,B) + \epsilon'] + (1 - \alpha^2)d(A,B) = d(A,B) + \alpha^2 \epsilon'$

which contradicts. Therefore, we can conclude that (2.1) holds.

Lemma- 15 Let T be an ICS mapping such that $T: X \to X$ also let A and B be nonempty subsets of a metric space X, (A,B) and (B,A) satisfy the property UC^*.

Let $F: A \times A \to B, G: B \times B \to A$ and (F,G) be a T-cyclic contraction. If $(x_0, y_0) \in A \times A$ and define

$$x_{n+1} = F(x_n, y_n),\ y_{n+1} = F(y_n, x_n)$$

and

$$x_{n+2} = G(x_{n+1}, y_{n+1}),\quad y_{n+2} = G(y_{n+1}, x_{n+1})$$

for all $n \in N \cup \{0\}$, then $\{x_n\}, \{y_n\}, \{x_{n+1}\}$ and $\{y_{n+1}\}$ are Cauchy sequences.

Proof: By Lemma-13, we have $d(x_n, x_{n+1}) \to d(A,B)$ and $d(x_{n+1}, x_{n+2}) \to d(A,B)$. Since (A,B) satisfies property UC, we get $d(x_n, x_{n+2}) \to 0$. Similarly, we also have $d(x_{n+1}, x_{n+3}) \to 0$ because (B,A) satisfies property UC.

We now show that for every $\epsilon > 0$ there exists $N \in \mathcal{N}$ such that

$$d(x_m, x_{n+1}) \leq d(A,B) + \epsilon \tag{2.2}$$

for all $m > n \geq N$

Suppose (2.2)not hold, then there exists $\epsilon > 0$ such that for all $k \in N$ there exists $m_k > n_k \geq k$ such that

$$d(Tx_{m_k}, Tx_{n_k+1}) > T(d(A,B)) + \epsilon. \tag{2.3}$$

Further, corresponding to n_k, we can choose m_k in such a way that it is the smallest integer with $m_k > n_k$ and satisfying above relation. Now we have

$$\begin{aligned} T(d(A,B)) + \epsilon &< d(Tx_{m_k}, Tx_{n_k+1}) \\ &\leq d(Tx_{m_k}, Tx_{m_k-2}) + d(Tx_{m_k-2}, Tx_{n_k+1}) \\ &\leq d(Tx_{2m_k}, Tx_{2m_k-2}) + T(d(A,B)) + \epsilon. \end{aligned}$$

Taking $k \to \infty$, we have $d(Tx_{2m_k}, Tx_{2n_k+1}) \to T(d(A,B)) + \epsilon$.

By Lemma 13, there exists $N \in \mathcal{N}$ such that

$$\frac{\alpha}{2}[d(Tx_{m_k}, Tx_{n_k+1}) + d(Ty_{m_k}, Ty_{n_k+1})] < T(d(A,B)) + \epsilon \tag{2.4}$$

for all $m > n \geq \mathcal{N}$. By using the triangle inequality, we get

$$\begin{aligned} T(d(A,B)) + \epsilon &< d(Tx_{m_k}, Tx_{n_k+1}) \\ &\leq d(Tx_{m_k}, Tx_{m_k+2}) + d(Tx_{m_k+2}, Tx_{n_k+3}) + d(Tx_{n_k+3}, Tx_{n_k+1}) \\ &= d(Tx_{m_k}, Tx_{m_k+2}) + d\left(TG(x_{m_k+1}, y_{m_k+1}), TF(x_{n_k+2}, y_{n_k+2})\right) + d(Tx_{n_k+3}, Tx_{n_k+1}) \\ &\leq d(Tx_{m_k}, Tx_{m_k+2}) + \frac{\alpha}{2}[d(x_{m_k+1}, x_{n_k+2}) + d(y_{m_k+1}, y_{n_k+2})] \\ &\qquad + (1 - \alpha)T(d(A,B)) + d(Tx_{n_k+3}, Tx_{n_k+1}) \\ &= \frac{\alpha}{2}\left[d\left(TF(x_{m_k}, y_{m_k}), TG(x_{n_k+1}, y_{n_k+1})\right)\right] + \frac{\alpha}{2}\left[d\left(TF(y_{m_k}, x_{m_k}), TG(y_{n_k+1}, x_{n_k+1})\right)\right] \\ &\qquad + (1 - \alpha)T(d(A,B)) + d(Tx_{m_k}, Tx_{m_k+2}) + d(Tx_{n_k+3}, Tx_{n_k+1}) \\ &\leq \frac{\alpha}{2}\left[\frac{\alpha}{2}[d(Tx_{m_k}, Tx_{n_k+1}) + d(Ty_{m_k}, Ty_{n_k+1}) + (1 - \alpha)T(d(A,B))]\right] \\ &\qquad + \frac{\alpha}{2}\left[\frac{\alpha}{2}[d(Ty_{m_k}, Ty_{n_k+1}) + d(Tx_{m_k}, Tx_{n_k+1}) + (1 - \alpha)T(d(A,B))]\right] \\ &\qquad + (1 - \alpha)T(d(A,B)) + d(Tx_{m_k}, Tx_{m_k+2}) + d(Tx_{n_k+3}, Tx_{n_k+1}) \end{aligned}$$

$$= \alpha^2 \left[d\left(Tx_{m_k}, Tx_{n_k+1}\right) + d\left(Ty_{m_k}, Ty_{n_k+1}\right)\right]$$
$$+ (1-\alpha^2)\, T(d(A,B)) + d\left(Tx_{m_k}, Tx_{m_k+2}\right) + d\left(Tx_{n_k+3}, Tx_{n_k+1}\right)$$
$$< \alpha^2\, T(d(A,B)) + \epsilon\,) + (1-\alpha^2)\, T(d(A,B)) + d(Tx_{m_k}, Tx_{m_k+2})$$
$$+ d(Tx_{n_k+3}, Tx_{n_k+1})$$
$$= \alpha^2 \epsilon + T\left(d(A,B)\right) + d\left(Tx_{m_k}, Tx_{m_k+2}\right) + d\left(Tx_{n_k+3}, Tx_{n_k+1}\right).$$

Taking $k \to \infty$, we get

$$T\left(d(A,B)\right) + \epsilon \le T\left(d(A,B)\right) + \alpha^2 \epsilon$$

which contradicts. Therefore, condition (2.2) holds. Since (2.2) holds and $d(Tx_n, Tx_{n+1}) \to T\left(d(A,B)\right)$, by using property UC^* of (A,B), we have $\{Tx_n\}$ is a Cauchy sequence. In similar way, we can prove that $\{Ty_n\}, \{Tx_{n+1}\}$ and $\{Ty_{n+1}\}$ are Cauchy sequences. Since T is ICS mapping, i.e T is injective mapping, we have $\{x_n\}$, $\{y_n\}, \{x_{n+1}\}$ and $\{y_{n+1}\}$ are a Cauchy sequences.

Here we state the main results of this article in the existence and convergence of coupled best proximity points for cyclic contraction pairs on nonempty subsets of metric spaces satisfying the property UC^*.

Theorem-16 Let T be an ICS mapping on X and A and B be nonempty closed subsets of a metric space X such that (A,B) and (B,A) have a property $UC^*, F\colon A\times A \to B$ and $G\colon B\times B \to A$ and let the ordered pair (F,G) is a T- cyclic contraction. If $(x_0, y_0) \in A \times A$ and define

$$x_{n+1} = F(x_n, y_n), \; y_{n+1} = F(y_n, x_n)$$

and

$$x_{n+2} = G(x_{n+1}, y_{n+1}), \; y_{n+2} = G(y_{n+1}, x_{n+1})$$

for all $n \in N \cup \{0\}$. Then F has a coupled best proximity point $(r,s) \in A^2$ and G has a coupled best proximity point $(r', s') \in B^2$.

Moreover, we have $x_n \to r, \; y_n \to s, \; x_{n+1} \to r', \; y_{n+1} \to s'$.

Furthermore, if $r = s$ and $r' = s'$, then

$$d(r, r') + d(s, s') = d(A,B).$$

Proof : By Lemma-13, we get $d(Tx_n, Tx_{n+1}) \to T\left(d(A,B)\right)$. Using Lemma-13, we have $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences. Thus, there exists $r, s \in A$ such that $Tx_n \to Tr, \; Ty_n \to Ts$.

We obtain that

$$T\left(d(A,B)\right) \le d(Tr, Tx_{n-1}) \le d(Tr, Tx_n) + d(Tx_n, Tx_{n-1}). \tag{2.5}$$

Letting $n \to \infty$ in (2.5), we have $d(Tr, Tx_{n-1}) \to T\left(d(A,B)\right)$. By a similar argument we also have

$$d(Ts, Ty_{n-1}) \to T\left(d(A,B)\right).$$

It follows that

$$d\left(Tx_n, TF(r,s)\right) = d\left(TG(x_{n-1}, y_{n-1}), TF(r,s)\right)$$
$$\le \frac{\alpha}{2}\left[d(Tx_{n-1}, Tr) + d(Ty_{n-1}, Ts)\right] + (1-\alpha)T(d(A,B)).$$

Taking $n \to \infty$, we get $d\left(Tr, TF(r,s)\right) = T(d(A,B))$. Similarly, we can prove that

$$d\left(Ts, TF(s,r)\right) = T\left(d(A,B)\right)$$

Since T is injective mapping.

Therefore, we have (r,s) is a coupled best proximity point of F.

In similar way, we can prove that there exists $r', s' \in B$ such that $Tx_{n+1} \to r'$ and $Ty_{n+1} \to s'$. Moreover, we have

$$d\left(Tr', TG(r', s')\right) = T\left(d(A,B)\right),$$

and

$$d\left(Ts', TF(s', r')\right) = T\left(d(A,B)\right)$$

and so (r', s') is a coupled best proximity point of G.

Finally, we assume that $r = s$ and $r' = s'$ and then we show that

$$d(Tr, Tr') + d(Ts, Ts') = 2T\big(d(A, B)\big).$$

For all $n \in N$, we obtain that

$$d(Tx_n, Tx_{n+1}) = d\big(TG(x_{n-1}, y_{n-1}), TF(x_n, y_n)\big)$$
$$\leq \frac{\alpha}{2}[\, d(Tx_{n-1}, Tx_n) + d(Ty_{n-1}, Ty_n)] + (1-\alpha)T\big(d(A, B)\big).$$

Letting $n \to \infty$, we have

$$d(Tr, Tr') \leq \frac{\alpha}{2}[d(Tr, Tr') + d(Ts, Ts')] + (1-\alpha)T(d(A, B)). \quad (2.6)$$

For all $n \in N$, we have

$$d(Ty_n, Ty_{n+1}) = d\big(TG(y_{n-1}, x_{n-1}), TF(y_n, x_n)\big)$$
$$\leq \frac{\alpha}{2}[\, d(Ty_{n-1}, Ty_n) + d(Tx_{n-1}, Tx_n)] + (1-\alpha)T(d(A, B)).$$

Letting $n \to \infty$, we have

$$d(Ts, Ts') \leq \frac{\alpha}{2}[\, d(Ts, Ts') + d(Tr, Tr')] + (1-\alpha)T(d(A, B)). \quad (2.7)$$

Similarly we can write,

It follows from (2.6) and (2.7) that

$$d(Tr, Tr') + d(Ts, Ts') \leq \frac{\alpha}{2}[d(Tr, Tr') + d(Ts, Ts')] + 2(1-\alpha)T(d(A, B))$$

which implies that

$$d(Tr, Tr') + d(Ts, Ts') \leq 2\,T(d(A, B)). \quad (2.8)$$

Since $T(d(A, B)) \leq d(Tr, Tr')$ and $T(d(A, B)) \leq d(Ts, Ts')$, we have

$$d(Tr, Tr') + d(Ts, Ts') \geq 2\,T(d(A, B)).$$

From (2.7)and (2.8),we get

$$d(Tr, Tr') + d(Ts, Ts') = 2\,T(d(A, B)). \quad (2.9)$$

Since T is injective mapping which implies

$$d(r, r') + d(s, s') = 2\,d(A, B). \quad (2.10)$$

This complete the proof.

Note that every pair of nonempty closed subsets A,B of a uniformly convex Banach space X such that A is convex satisfies the property UC.

Therefore, we obtain the following corollary.

Corollary- 17 Let T be an ICS mapping such that $T: X \to X$ and A and B be nonempty closed convex subsets of a uniformly convex Banach space $X, F: A \times A \to B$ and $G: B \times B \to A$ and let the ordered pair (F,G) be a T- cyclic contraction. Let $(x_0, y_0) \in A \times A$ and define

$$x_{n+1} = F(x_n, y_n),\ x_{n+2} = G(x_{n+1}, y_{n+1}), \quad \text{and } y_{n+1} = F(y_n, x_n),\ y_{n+2} = G(y_{n+1}, x_{n+1})$$

for all $n \in N \cup \{0\}$. Then F has a coupled best proximity point $(r, s) \in A \times A$ and G has a coupled best proximity point $(r', s') \in B \times B$.

Moreover, we have $x_n \to r,\ y_n \to s,\ x_{n+1} \to r',\ y_{n+1} \to s'$.

Furthermore, if $r = s$ and $r' = s'$, then

$$d(r, r') + d(s, s') = 2\,d(A, B).$$

Next, we give some illustrative example of Corollary 17.

Example- 18 Consider uniformly convex Banach space $X = R$ with the usual norm. Let $A = [1,2]$ and $B = [-1, -2]$. Thus $d(A, B) = 2$. Define $F: A \times A \to B$ and $G: B \times B \to A$ by

$$F(x,y) = \frac{-2x-3y-1}{6} \quad \text{and} \quad G(x,y) = \frac{-2x-3y+1}{6}.$$

For arbitrary $(x,y) \in A \times A$ and $(u,v) \in B \times B$ and fixed $p = \frac{1}{3}$ and $q = \frac{1}{2}$ we get

$$d(F(x,y), G(u,v)) = \left|\frac{-x-y-1}{6} - \frac{-u-v+1}{6}\right|$$
$$\leq \frac{2|x-u|+3|y-v|}{6} + \frac{1}{3}$$
$$= \frac{1}{3}d(x,u) + \frac{1}{2}d(y,v) + \big(1-(p+q)\big)d(A,B)$$

This implies that (F,G) is a cyclic contraction with $\alpha = \frac{1}{2}$. Since A and B are closed convex, we have (A,B) and (B,A) satisfy the property UC^*.

Therefore, all hypothesis of Corollary 17 hold. So F has a coupled best proximity point and G has a coupled best proximity point. We note that a point $(1,1) \in A \times A$ is a unique coupled best proximity point of F and a point $(-1,-1,) \in B \times B$ is a unique coupled best proximity point of G. Furthermore, we get

$$d(1,-1) + d(1,-1) = 4 = 2d(A,B).$$

Next, we give the coupled best proximity point result in compact subsets of metric spaces.

Theorem- 19 Let T be an ICS mapping such that $T: X \to X$ and A and B be nonempty compact subsets of a metric space X, $F: A \times A \to B$ and $G: B \times B \to A$ and let the ordered pair (F,G) be a cyclic contraction. Let $(x_0, y_0) \in A \times A$ and define

$$x_{n+1} = F(x_n, y_n),\ x_{n+2} = G(x_{n+1}, y_{n+1})$$
$$y_{n+1} = F(y_n, x_n),\ y_{n+2} = G(y_{n+1}, x_{n+1})$$

for all $n \in N \cup \{0\}$. Then F has a coupled best proximity point $(r,s) \in A \times A$ and G has a coupled best proximity point $(r', s') \in B \times B$.

Moreover, we have $x_n \to r,\ y_n \to s,\ x_{n+1} \to r',\ y_{n+1} \to s'$.

Furthermore, if $r = s$ and $r' = s'$, then

$$d(r,r') + d(s,s') = 2\, d(A,B).$$

Proof : Since $x_0, y_0 \in A$ and

$$x_{n+1} = F(x_n, y_n),\ x_{n+2} = G(x_{n+1}, y_{n+1})$$
$$y_{n+1} = F(y_n, x_n),\ y_{n+2} = G(y_{n+1}, x_{n+1})\,]$$

for all $n \in N \cup \{0\}$, we have $x_n, y_n \in A$ and $x_{n+1}, y_{n+1} \in A$ for all $n \in N \cup \{0\}$. As A is compact, the sequences $\{x_n\}$ and $\{y_n\}$ have convergent subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ respectively, such that

$$x_{n_k} \to r \in A,\ y_{n_k} \to s \in A.$$

Now, we have

$$T(d(A,B)) \leq d(Tr, Tx_{n_k-1}) \leq d(Tr, Tx_{n_k}) + d(Tx_{n_k}, Tx_{n_k-1}) \qquad (2.11)$$

By Lemma-13, we have $d(Tx_{n_k}, Tx_{n_k-1}) \to T(d(A,B))$.

Taking $k \to \infty$ in (2.11), we get

$$d(Tr, Tx_{n_k-1}) \to T(d(A,B)).$$

By a similar argument we observe that

$$d(Ts, Tx_{n_k-1}) \to T(d(A,B)).$$

Note that

$$T(d(A,B)) \leq d\left(Tx_{n_k}, TF(r,s)\right) = d\left(TG(x_{n_k-1}, y_{n_k-1}), TF(r,s)\right)$$
$$\leq \frac{\alpha}{2}\left[d(Tx_{n_k-1}, Tr) + d(Ty_{n_k-1}, Ts)\right] + (1-\alpha)T(d(A,B)).$$

Taking $k \to \infty$, we get $d(Tr, TF(r,s)) = T(d(A,B))$. Similarly, we can prove that

$$d(Ts, TF(s,r)) = T(d(A,B)).$$

Thus F has a coupled best proximity $(r,s) \in A \times A$. In similar way, since B is compact, we can also prove that G has a coupled best proximity point $(r',s') \in B \times B$. For

$$d(Tr,Tr') + d(Ts,Ts') = 2T\big(d(A,B)\big)$$

Since T is injective mapping. So we have

$$d(r,r') + d(s,s') = 2d(A,B)$$

similar to the final step of the proof of Theorem-16.

This complete the proof.

3. Coupled Fixed Point Theorems

In this section, we give the new coupled fixed point theorem for a cyclic contraction pair.

Theorem- 20 Let T be an ICS mapping such that $T: X \to X$ also A and B be nonempty closed subsets of a metric space $X, F: A \times A \to B$ and $G: B \times B \to A$ and let the ordered pair (F,G) be a T-cyclic contraction. Let $(x_0, y_0) \in A \times A$ and define

$$x_{n+1} = F(x_n, y_n),\ x_{n+2} = G(x_{n+1}, y_{n+1})$$
$$y_{n+1} = F(y_n, x_n),\ y_{n+2} = G(y_{n+1}, x_{n+1})$$

for all $n \in N \cup \{0\}$. If $d(A,B) = 0$, then F has a coupled fixed point $(r,s) \in A \times A$

and G has a coupled fixed point $(r',s') \in B \times B$.

Moreover, we have $x_n \to r,\ \ y_n \to s,\ \ x_{n+1} \to r',\ \ y_{n+1} \to s'$.

Furthermore, if $r = r'$ and $s = s'$, then F and G have a common coupled fixed point in $(A \cap B)^2$.

Proof : Since $d(A,B) = 0$, we get (A,B) and (B,A) satisfy the property UC.

Therefore, by Theorem- 16, claim that F has a coupled best proximity point $(r,s) \in A \times A$ that is

$$d(Tr, TF(r,s)) = d(Ts, TF(s,r)) = T(d(A,B)) \quad (3.1)$$

and G has a coupled best proximity point $(r',s') \in B \times B$ that is

$$d(Tr', TG(r',s')) = d(Ts', TG(s',r')) = T(d(A,B)). \quad (3.2)$$

From (3.1) and $d(A,B) = 0$, we conclude that

$$r = F(r,s),\ \ s = F(s,r).$$

that is (r,s) is a coupled fixed point of F. It follows from (3.2) and $d(A,B) = 0$, we get

$$r' = G(r',s'),\ \ and\ \ s' = G(s',r')$$

that is (r',s') is a coupled fixed point of G.

Next, we assume that $r = r'$ and $s = s'$ and then we show that

F and G have a unique common coupled fixed point in $(A \cap B)^2$.

From Theorem-16, we get

$$d(Tr,Tr') + d(Ts,Ts') = 2\,T(d(A,B)). \quad (3.3)$$

Since $T(d(A,B)) = 0$, we get

$$d(Tr,Tr') + d(Ts,Ts') = 0$$

Since T is injective mapping.

which implies that $r = r'$ and $s = s'$.

Therefore, we conclude that $(r,s) \in (A \cap B)^2$ is common coupled fixed point of F and G.

Example- 21 Consider $X = R$ with the usual metric, $A = [-2,0]$ and $B = [0,2]$. Define $T: X \to X, F: A \times A \to B$ and $G: B \times B \to A$ by $Tx = \frac{x}{4}$

$$F(x,y) = -\frac{2x+2y}{5} \text{ and } G(u,v) = -\frac{2u+2v}{5}.$$

Then $d(A,B) = 0$ and (F,G) is a T- cyclic contraction with $\alpha = \frac{4}{5}$.

Indeed, for arbitrary $(x,y) \in A \times A$ and $(u,v) \in B \times B$, we have

$$d(F(x,y),G(u,v)) = \left|-\frac{2x+2y}{5}+\frac{2u+2v}{5}\right|$$
$$\leq \frac{\alpha}{2}\,[d(Tx,Tu) + d(Ty,Tv)] + (1-\alpha)T\big(d(A,B)\big).$$

Therefore, all hypothesis of Theorem-20 hold. So F and G have a common coupled fixed point and this point is $(0,0) \in (A \cap B)^2$.

If we take $A = B$ in Theorem 20, then we get the following results.

Corollary- 22 Let T be an ICS mapping such that $T: X \to X$ and A be a nonempty closed subset of a complete metric space $X, F: A \times A \to A$ and $G: A \times A \to A$ and let the ordered pair (F,G) be a T-cyclic contraction. Let $(x_0, y_0) \in A \times A$ and define

$$x_{n+1} = F(x_n, y_n),\ y_{n+1} = F(y_n, x_n) \quad \text{and} \quad x_{n+2} = G(x_{n+1}, y_{n+1}),\ y_{n+2} = G(y_{n+1}, x_{n+1})$$

for all $n \in N \cup \{0\}$.

Then F has a coupled fixed point $(r,s) \in A \times A$

and G has a coupled fixed point $(r', s') \in B \times B$.

Moreover, we have $x_n \to r,\ y_n \to s,\ x_{n+1} \to r',\ y_{n+1} \to s'$.

Furthermore, if $r = r'$ and $s = s'$, then F and G have a common coupled fixed point in A× A.

We take F = G in Corollary 22, then we get the following results

Corollary- 23 Let T be an ICS mapping such that $T: X \to X$ and A be nonempty closed subsets of a complete metric space X, $F: A \times A \to A$ and

$$d(TF(x,y), TF(u,v)) \leq \frac{\alpha}{2}\,[d(Tx,Tu) + d(Ty,Tv)]$$

for all $(x,y),(u,v) \in A \times A$. Then F has a coupled fixed point $(r,s) \in A \times A$.

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