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Linear operators on a Hilbert space and some of the related properties.

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ABSTRACT

We study Birkhoff-James orthogonality and isosceles orthogonality of bounded linear operators between Hilbert spaces and Banach spaces. We explore Birkhoff-James orthogonality of bounded linear operators in light of a new notion in-troduced by us and also discuss some of the possible applications in this regard. We also study isosceles orthogonality of bounded (positive) linear operators on a Hilbert space and some of the related properties, including that of operators having disjoint support. We further explore the relations between Birkhoff-James orthogonality and isosceles orthogonality in a general Banach space.

Keywords: Hilbert spaces, Linear operators.

INTRODUCTION

The primary purpose of the present paper is to explore orthogonality of bounded linear operators between Hilbert spaces and Banach spaces. Unlike the Hilbert space case, there is no universal notion of orthogonality in a Banach space. However, it is possible to have several notions of orthogonality in a Banach space, each of which generalizes some particular aspect of Hilbert space orthogonality. Indeed, one of the root causes of the vast differences between the geometries of Hilbert spaces and Banach spaces is the lack of a standard orthogonality notion in the later case. On the other hand, this makes the study of orthogonality of bounded linear operators, between Hilbert spaces and Banach spaces, an interesting and deeply rewarding area of research. In recent times, several authors have explored this topic and obtained many interesting results involving orthogonality of bounded linear operators. In this paper, among other things, we extend, improve and generalize some of the earlier results on orthogonality of bounded linear operators. Without further ado, let us first establish our notations and terminologies to be used throughout the paper.

Letters X, Y denote Banach spaces, over the field $K \in \{R, C\}$. Let $B_X = \{x \in X : x \leq 1\}$ and $S_X = \{x \in X : x = 1\}$ be the unit ball and the unit sphere of X respectively. Let B(X, Y) and K(X, Y) denote the Banach space of all bounded linear operators and compact operators from X to Y respectively, endowed with the usual operator norm. We write B(X, Y) = B(X) and K(X, Y) = K(X) if X = Y. The symbol I_X stands for the identity operator on X. We omit the suffix in case there is no confusion. We reserve the symbol H for a Hilbert space over the field K.

Throughout the paper, we consider only separable Hilbert spaces. In this paper, mostly in the context of bounded linear operators, we discuss three of the most important orthogonality types in a Banach space, namely, Birkhoff-James orthogonality, isosceles orthogonality and Roberts orthogonality . Let us first state the relevant definitions.

Definition 1.1. For any two elements x, y ∈ X, we say that x is Birkhoff-James or-thogonal to y, written as $x \perp_B y$, if for all $\lambda \in K$, the following holds:

$$x \leq x + \lambda y .$$

Definition 1.2. For any two elements x, y ∈ X, we say that x is isosceles orthogonal to y, written as $x \perp_I y$, if the following holds:

$$x + y = x - y .$$

For complex Banach spaces, one needs to consider the following relation for isosceles orthogonality:

$x \perp_I y \Leftrightarrow$ ($x + iy$ $= x - iy$. (1.3)

$$x + y = x - y$$

Definition 1.3. For any two elements x, y ∈ X, we say that x is Roberts orthogonal to y, written as $x \perp_R y$, if for all $\lambda \in K$, the following holds:

$$x + \lambda y = x - \lambda y .$$

In order to have a better description of Birkhoff-James orthogonality of bounded linear operators between Banach spaces, we introduce the following notation for any T, A ∈ B(X, Y):

$O_{T,A} = \{x \in S_X : Tx \perp_B Ax\}$. Given T ∈ B(X, Y), define the norm attainment set of T as

$$M_T = \{x \in S_X : \quad Tx = T \} .$$

In order to study the properties of the set $O_{T,A}$, in the context of a real Banach space, we require the following notions introduced in [8]. .

Definition 1.4. Let X be a real or complex normed space. Let x, y ∈ X . We say that $y \in x^+$ if $x + \lambda y \geq x$ for all $\lambda \geq 0$. Accordingly, we say that $y \in x^-$ if $x + \lambda y \geq x$ for all $\lambda \leq 0$.

Let X be a real or complex normed space and x, y ∈ X . We say that x is r-orthogonal to y, denoted by $x \perp^r_B y$ if $y \in x^+$ and $y \in x^-$.

In the context of T ∈ B(H), the corresponding norm attainment set M_T was com-pletely characterized in [7]. We would like to remark that Birkhoff-James orthogonal-ity of bounded linear operators on a finite-dimensional Hilbert space H was completely characterized by Bhatia and Semrl in [5]:

For T, A ∈ B(H), $T \perp_B A \Longleftrightarrow O_{T,A} \cap M_T = \emptyset$.

The notion of Birkhoff-James orthogonality is intimately connected with the notion of smoothness in Banach spaces. A non-zero element x ∈ X is said to be a smooth point if there exists a unique norm one functional $f \in X^*$ such that f (x) = x . It is easy to observe that the notion of smoothness is meaningful in the space of bounded linear operators between Banach spaces.

For $A \in B(H)$, we use the notations A^*, $R(A)$, $N(A)$, to denote the adjoint, the range and the kernel of A respectively. If A, B are self-adjoint elements of B(H) we write $A \leq B$ whenever $Ax, x \leq Bx, x$ for all $x \in H$. An element $A \in B(H)$ such that $A \geq 0$ is called positive. For every $A \geq 0$, there exists a unique positive $A^{1/2} \in B(H)$ such that $A = (A^{1/2})^2$. For any $B \subseteq B(H)$, B^+ denotes the subset of all positive operators of B.

Birkhoff-james orthogonality of bounded linear operators

We begin this section by obtaining an extension of the finite-dimensional Bhatia-Semrl theorem to the infinite-dimensional setting, with certain additional assumptions. We would like to remark that such an extension was obtained by W´ojcik in [2], in the context of real Hilbert spaces, without any other assumption. However, we present the following result in the context of smooth compact operators on a reflexive real Banach space.

Theorem Let X be a reflexive real Banach space and Y be any real Banach space. Let $T, A \in K(X, Y)$. Let us further assume that T is smooth in B(X, Y). Then $T \perp_{\mathbf{B}} A$ if and only if $O_{\mathbf{T,A}} \cap M_{\mathbf{T}} = \emptyset$.

Proof. The sufficient part of the theorem is trivially true. Let us prove the necessary part. Since X is reflexive and T is compact, it follows that $M_{\mathbf{T}} = \emptyset$. As T is smooth in B(X, Y),

Let us now study the set $O_{\mathbf{T,A}}$, when $T, A \in B(X, Y)$ are given. First we obtain an easy sufficient condition for Birkhoff-James orthogonality of two bounded linear operators T, A in terms of the set $O_{\mathbf{T,A}}$. We would like to note that the following theorem implies that unless $T \perp_{\mathbf{B}} A$, $O_{\mathbf{T,A}}$ cannot be the whole of S_X.

Orthogonality in B(H)

We begin this section by proving that in the context of bounded linear operators on a Hilbert space, disjoint support implies both Birkhoff-James orthogonality and isosceles orthogonality. In particular, it follows from our next result that for operators having disjoint support, Birkhoff-James orthogonality relation is symmetric.

Proposition Let $A, B \in B(H)$ with disjoint support, then the following holds:

(1) $A \perp_{\mathbf{B}} B$ and $B \perp_{\mathbf{B}} A$.

(2) $A \perp_{\mathbf{I}} B$.

Proof. (1) We observe that for any $\lambda \in C$,

$$|A + \lambda B|^2 = (A + \lambda B)^* (A + \lambda B) = (A^* + \overline{\lambda} B^*)(A + \lambda B)$$
$$= |A|^2 + A^* \lambda B + \overline{\lambda} B^* A + |\lambda|^2 |B|^2 \geq |A|^2 .$$

Therefore, using Lemma, it follows that

$$A + \lambda B \geq A \quad \text{for all } k \in C.$$

Similarly, we obtain that $|B + \mu A|^2 \geq |B|$ for all $\mu \in C$. This completes the proof of the first part of the proposition.

(2) If $A, B \in B(H)$ have disjoint support then $A \perp_{\mathbf{I}} B$, since

$$A + B^2 = \sup\{ (A + B)h^2 : h \in S_H\}$$
$$= \sup\{ Ah^2 + Bh^2 : h \in S_H\} = A - B^2.$$

This completes the proof of the second part of the proposition and establishes it completely.

Relations between different types of orthogonality

In this short section we study the relations between Birkhoff-James orthogonality and isosceles orthogonality. Before proceeding any further, let us mention the following fact that serves as a motivation behind our exploration in this section.

Bottazzi et. al. studied the equivalence of Birkhoff-James orthogonality and isosceles orthogonality of positive operators A, B in a p−Schatten ideal in [8]. Indeed, for every A, B ∈ $B(H)^+$ and $1 < p \leq 2$, they proved that

$$A \perp^p_B B \Leftrightarrow A \perp^p_I B.$$

However, it is not difficult to observe that there are many examples in B(H) (and more general in any Banach space) that show $\perp_B$ and $\perp_I$ are independent orthogonality types and none of them imply the other. Our purpose in this section is to establish relations between these two orthogonality types, in the sense that we determine which additional conditions may be required to have "$\perp_B \Rightarrow \perp_I$" and vice versa.

Proposition 1. Let X be a Banach space and x, y ∈ X. Let us further assume that $(x + y)\perp_B y$ and $(x - y)\perp_B y$. Then $x \perp_I y$.

Proof. By the hypothesis, we have,

$(x + y)\perp_B y \Rightarrow \quad x + y \leq \quad x + y + \lambda y \ \forall \ \lambda \in K.$

Taking $\beta = 1 + \lambda$, we have, $\quad x + y \leq \quad x + \beta y$. In particular for $\beta = -1$, we get $x + y \leq \ x - y$. Analogously, from the hypothesis $(x - y)\perp_B y$ and we obtain $x-y \leq \ x+y$. This proves that $x \perp_I y$ and completes the proof of the proposition.

In order to address the converse question, we introduce the concept of strongly isosce-les orthogonality in real Banach spaces.

Definition 2. Let X be a real or complex normed space and x, y ∈ X . We say that x is strongly isosceles orthogonal to y, written as $x \perp_{SI} y$ if

(i) $x \perp_I y$.

(ii) there exists a real sequence $\{\lambda_n\}_{n\in N}$, with $\lambda_n > 0$, such that $\lim_{n\to\infty} \lambda_n = 0$ and $x \perp_I \lambda_n y$ for all n ∈ N.

the definition of strongly isosceles orthogonality because we are trying to address the question that asks Isosceles orthogonality, along with which additional conditions, implies Birkhoff-James orthogonality.

We have already discussed that Roberts orthogonality is stronger and restrictive than either of Birkhoff-James orthogonality and isosceles orthogonality. Moreover, it is ob-vious that $A \perp_R B \Rightarrow A \perp_{SI} B$. In the examples we show that the converse of this statement is not necessarily true. We deliberately give the examples using different norms on B(H), to make them more illustrative.

REFERENCES

[1] J. Alonso, H. Martini, and S. Wu. On birkhoff orthogonality and isosceles orthogonality in normed linear spaces. Aequationes Math., 83(1-2):153–189, 2012.

[2] J. Arazy. The isometries of C_p. Israel J. Math., 22(3-4):247–256, 1975.

[3] C. Ben´ıtez, M. Fern´andez, and M. L. Soriano. Orthogonality of matrices. Linear Algebra Appl., 422(1):155–163, 2007.

[4] R. Bhatia. Matrix analysis, volume 169 of Graduate Texts in Mathematics. Springer-Verlag, New

York, 1997.

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[5] R. Bhatia and P. Semrl. Orthogonality of matrices and some distance problems. Linear Algebra Appl., 287(1-3):77–85, 1999. Special issue celebrating the 60th birthday of Ludwig Elsner.

[6] T. Bhattacharyya and P. Grover. Characterization of Birkhoff-James orthogonality. J. Math. Anal.

Appl., 407(2):350–358, 2013.

[7] G. Birkhoff. Orthogonality in linear metric spaces. Duke Math. J., 1(2):169–172, 1935.

[8] T. Bottazzi, C. Conde, M. S. Moslehian, and A. Zamani. Orthogonality and parallelism of operators on various banach spaces. J. Aus. Math. Soc., to appear, 2018.