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Commutativity of prime near rings with derivations

[1]Leta Hailu, [2]Adnew Markos, [3]Arbo Nigusie, [4]Temam Kesim

[1234]Lecturer, Department of Mathematics

[1]Arsi University, Assela, Ethiopia, [2]Jimma University, Jimma, Ethiopia

[3]Oda Bultum University, Chiro, Ethiopia, [4]Jigjiga University, Jigjiga, Ethiopia

Abstract Let N be a near ring. An additive mapping $d: N \to N$ is said to be a derivation on N if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ or equivalently $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. The purpose of the present paper is to prove some theorems in the setting of a semigroup ideal of a 3-prime near ring admitting a derivation, thereby extending some identities on derivations.

Keywords: prime near ring, derivations, semigroup ideal.

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1. INTRODUCTION

Throughout the paper, N denotes a zero-symmetric right near ring with multiplicative centre Z; and for any pair of elements $x, y \in N$, $[x, y]$ denotes the commutator $xy - yx$ while the symbol (x, y) denotes the additive commutator $x + y - x - y$. An element x of N is said to be distributive if $x(y + z) = xy + xz$, for all $y, z \in N$. A near ring N is called zero-symmetric if $x0 = 0$, for all $x \in N$ (recall that right distributivity yields that $0x = 0$). The near ring N is said to be 3-prime if $xNy = \{0\}$ for $x, y \in N$ implies that $x = 0$ or $y = 0$. A near ring N is called 2-torsion free if $(N, +)$ has no element of order 2. A nonempty subset U of N is called a semigroup right (resp. semigroup left) ideal if $UN \subseteq U$ (resp. $NU \subseteq U$) and if U is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal. An additive mapping $d: N \to N$ is a derivation on N if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ or equivalently as noted in [10], that $d(xy) = d(x)y + xd(y)$, for all $x, y \in N$.

The recent literature contains numerous results on commutativity in prime and semi-prime rings admitting suitably constrained derivations and generalized derivations, and several authors have proved comparable results on near-rings. In fact, the relationship between the commutativity of a 3-prime near-ring N and the behavior of a derivation on N was initiated in 1987 by Bell and Mason [7]. In [8], M. Ashraf and et al. generalize some of their results by assuming that the commutativity condition is imposed on a near-ring. In this paper, we will study the commutativity theorems for the 3-prime near-ring by treating the case of derivations satisfying certain algebraic identities involving semigroup ideals. Some of our results, which deal with conditions on semigroup ideals, extend earlier commutativity results involving similar conditions on near-rings.

2. Preliminary Results

We begin with the several lemmas, most of which are known. Those for which neither a proof nor a precise citation is given are to be found in [3]. Similar results can be obtained for right near ring.

Lemma 2.1 Let d be an arbitrary derivation of a near ring N. Then N satisfies the following partial distributive laws.
(i) $z(xd(y)+d(x)y)=zxd(y)+zd(x)y$, for all $x,y,z\in N$.
(ii) $z\big(d(x)y+xd(y)\big)=zd(x)y+zxd(y)$, for all $x,y,z\in N$.

Lemma 2.2 Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Let d be a nonzero derivation on N.
(i) If $x,y\in N$ and $xUy=\{0\}$, then $x=0$ or $y=0$.
(ii) If $x\in N$ and $xU=\{0\}$ or $Ux=\{0\}$, then $x=0$.
(iii) $d(U)\neq\{0\}$.
(iv) If $x\in N$ and $d(U)x=\{0\}$ or $xd(U)=\{0\}$, then $x=0$.

Lemma 2.3 Let N be a 3-prime near ring. If Z contains a nonzero semigroup left ideal or a nonzero semigroup right ideal, then N is a commutative ring.

Lemma 2.4 Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. If d is a nonzero derivation of N such that $d(U)\subseteq Z$, then N is a commutative ring.

Lemma 2.5 Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. If d is a nonzero derivation of N such that $d([x,y])=0$ for all $x,y\in U$, then N is a commutative ring.

Proof By the hypothesis, we have

$$d(xy)=d(yx)\ \text{ for all }\ x,y\in U. \tag{2.1}$$

(2.1) can be written as

$$xd(y)+d(x)y=yd(x)+d(y)x\ \text{ for all } x,y\in U. \tag{2.2}$$

Replacing y by yx in (2.2), we obtain

$$xd(yx)+d(x)yx=yxd(x)+d(yx)x\ \text{ for all } x,y\in U. \tag{2.3}$$

Using (2.2), we get

$$xyd(x)+(yd(x)+d(y)x)x=yxd(x)+yd(x)x+d(y)x^2\ \text{ for all } x,y\in U.$$

This implies that

$$xyd(x)=yxd(x)\ \text{ for all } x,y\in U. \tag{2.4}$$

Again replacing y by zy for $z\in N$ in (2.4) and use (2.4), to get

$$\begin{aligned} xzyd(x)&=zyxd(x)\\ &=zxyd(x)\end{aligned}$$

This implies that

$$[x,z]yd(x)=0 \text{ for all } x,y\in U, z\in N.$$

-i.e.,

$$[x,z]Ud(x)=\{0\} \text{ for all } x,y\in U, z\in N.$$

Using Lemma 2.2 (i), we obtain $d(x)=0$ or $[x,z]=0$ for all $x\in U, z\in N$. In the former case $d(x)\in Z(N)$ as $0\in Z(N)$. In the later case $x\in Z(N)$, which yields that $d(x)\in Z(N)$. Thus in the both the cases we obtain $d(x)\in Z(N)$ for $x\in U$. - i.e., $d(U)\subseteq Z(N)$. Hence by Lemma 2.4, we get N is a commutative ring.

Lemma 2.6 Let N be a 3-prime near ring.

(i) If $z\in Z\setminus\{0\}$, then z is not a zero divisor.

(ii) If $z\in Z\setminus\{0\}$ and x is an element of N for which $xz\in Z$, then $x\in Z$.

(iii) If x is an element of N which centralizes some nonzero semigroup right ideal, then $x\in Z$.

Lemma 2.7 [**6**, **Lemma 1.8**] Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. If d is a derivation on N such that $d^2(U)=\{0\}$, then $d=0$.

3. Main Result

Theorem 3.1 Let N be a prime near ring and U be a nonzero semigroup ideal of N. If N admits a nonzero derivation d, then the following assertions are equivalent

(i) $d([x,y])=[d(x),y]$ for all $x,y\in U$.

(ii) $[d(x),y]=[x,y]$ for all $x,y\in U$.

(iii) N is a commutative ring.

Proof It is easy to verify that $(iii)\Rightarrow$ (i) and $(iii)\Rightarrow$ (ii).

$(i)\Rightarrow(iii)$ Assume that $d([x,y])=[d(x),y]$ for all $x,y\in U$. Replacing y by yx, we get $[d(x),yx]=d([x,y]x)$, for all $x,y\in U$. By definition of d above relation reduce to

$$xyd(x)=yd(x)x, \text{ for all } x,y\in U. \tag{3.1}$$

Substituting yz for y in above, we obtain $[x,y]zd(x)=0$ for all $x,y,z\in U$. This implies that $[x,y]Ud(x)=\{0\}$, using Lemma 2.2 (i), we obtain $[x,y]=0$ or $d(x)=0$ for all $x,y\in U$.

If there exist $x_o\in U$ such that $[x_o,y]=0$ for all $y\in U$, then using our assumption we get $[d(x_o),y]=0$ for all $y\in U$. In all cases, we arrive at $d(x)\in Z(N)$ for all $x\in U$ and Lemma 2.4 forces that N is a commutative ring.

$(ii)\Rightarrow(iii)$ Assume that $[d(x),y]=[x,y]$ for all $x,y\in U$. Replacing x by xy, we get $[d(xy),y]=[xy,y]$, for all $x,y\in U$. This implies that $[d(xy),y]=[x,y]y=[d(x),y]y$ for all $x,y\in U$. In view of Lemma 2.1 partial distributive law, the last equation can be rewritten as

$d(x)y^2+xd(y)y-yxd(y)-yd(x)y=d(x)y^2-yd(x)y$, so that

$$xd(y)y=yxd(y) \text{ for all } x,y\in U. \tag{3.2}$$

Since Eq. (3.2) is same as Eq. (3.1), arguing as in the proof of $(i) \Rightarrow (iii)$ we find that N is a commutative ring.

Theorem 3.2 Let N be a 3-prime near ring and U be a nonzero additive subgroup and a semigroup ideal of N. If N admits a nonzero derivation d, then the following assertions are equivalent

(i) $d([x,y]) \in Z(N)$ for all $x, y \in U$.

(ii) N is a commutative ring.

Proof It is clear that $(ii) \Rightarrow$ (i).

$(i) \Rightarrow$ (ii). We are given that

$$d([x,y]) \in Z(N) \text{ for all } x, y \in U. \quad (3.3)$$

(a) If $Z(N) = \{0\}$, it follows $d([x,y]) = 0$, for all $x, y \in U$. By Lemma 2.5, we conclude that N is a commutative ring.

(b) If $Z(N) \neq \{0\}$, replacing y by yz in (3.3), where $z \in Z(N)$, we get

$$d([x,y])z + [x,y]d(z) \in Z(N) \text{ for all } x, y \in U, z \in Z(N). \quad (3.4)$$

Using (3.3) together with Lemma 2.1 (ii) Eq. (3.4) implies

$$[x,y]d(z) \in Z(N) \text{ for all } x, y \in U, z \in Z(N).$$

Accordingly, $0 = \big[[x,y]d(z), t\big] = \big[[x,y], t\big]d(z)$ for all $t \in N$ and thus

$$\big[[x,y], t\big]Nd(z) = \{0\} \text{ for all } x, y \in U, t \in N, z \in Z(N). \quad (3.5)$$

Using the primeness of N, from (3.5) it follows that $d\big(Z(N)\big) = \{0\}$ or $\big[[x,y], t\big] = 0$ for all $x, y \in U$ and $t \in N$.

Assuming that $\big[[x,y], t\big] = 0$ for all $x, y \in U$ and $t \in N$; substituting yx for y we get $\big[[x,y]x, t\big] = 0$ and therefore $[x,y][x,t] = 0$ for all $x, y \in U$ and $t \in N$. As $[x,y] \in Z(N)$,
hence

$$[x,y]N[x,y] = \{0\} \text{ for all } x, y \in U. \quad (3.6)$$

In the primeness of N. Equation (3.6) shows that $[x,y] = 0$ for all $x, y \in U$. Replacing y by yr for $r \in N$, we obtain $U \subseteq Z$. Lemma 2.3, force that N is a commutative ring.

On the other hand $d\big(Z(N)\big) = \{0\}$, then $d^2([x,y]) = 0$ for all $x, y \in U$. Replacing y by yx, we have

$$0 = d^2([x,xy]) = d^2([x,y])x + 2d([x,y])d(x) + [x,y]d^2(x) \text{ for all } x, y \in U.$$

Hence,

$$2d([x,y])d(x) + [x,y]d^2(x) \text{ for all } x, y \in U. \quad (3.7)$$

Taking $[u,v]$ instead of x in (3.7), and using the hypothesis 2-torsion freeness, we have

$$d([u,v])Nd\big([[u,v],y]\big) = \{0\}, \text{ for all } u, v, y \in U. \quad (3.8)$$

Since N is 3-prime, then (3.8) shows that

$$d([u,v]) = 0 \text{ or } d\big([[u,v],y]\big) = \{0\} \text{ for all } u, v, y \in U. \quad (3.9)$$

If there are two element $u_o, v_o \in U$ such that $d\big([[u_o,v_o],y]\big) = 0$ for all $y \in U$

$$[u_o,v_o]d(y) = d(y)[u_o,v_o] \text{ for all } y \in U. \quad (3.10)$$

Replacing y by $[u_o, v_o]t$ for $t \in N$ in the left hand side of equation (3.10) and using Lemma 2.1 (ii), we have $[u_o, v_o]d([u_o, v_o]t) = [u_o, v_o]d([u_o, v_o])t + [u_o, v_o]^2 d(t) = d([u_o, v_o])[u_o, v_o]t + [u_o, v_o]^2 d(t)$. Taking $[u_o, v_o]t$ for $t \in N$ instead of y in the right hand side of equation (3.10) and using (3.10), we get $d([u_o, v_o]t)[u_o, v_o] = d([u_o, v_o])t[u_o, v_o] + [u_o, v_o]^2 d(t)$. Comparing the last two equations, we get

$$d([u_o, v_o])N\big[[u_o, v_o], t\big] = \{0\} \text{ for all } t \in N.$$

Since N is 3-prime, we find that

$$d([u_o, v_o]) = 0 \text{ or } \big[[u_o, v_o], t\big] = 0 \text{ for all } t \in N. \tag{3.11}$$

Taking (3.11), into account, (3.9) implies that

$$d([u, v]) = 0 \text{ or } \big[[u, v], t\big] = 0 \text{ for all } u, v \in U, t \in N \tag{3.12}$$

If there are two element $u_1, v_1 \in U$ such that $\big[[u_1, v_1], t\big] = 0$ for all $t \in N$, then $[u_1, v_1] \in Z(N)$. By the hypothesis, we get $d([u_1, v_1]) = 0$, from (3.12) and the last expression, we conclude that

$$d([u, v]) = 0 \text{ for all } u, v \in U.$$

Finally, N is a commutative ring by Lemma 2.5.

Theorem 3.3 Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. If N admits a nonzero derivation d, then the following assertions are equivalent

(i) $[d(x), y] \in Z(N)$ for all $x, y \in U$.

(ii) N is a commutative ring.

Proof It is easy to see $(ii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Assume that

$$[d(x), y] \in Z(N) \text{ for all } x, y \in U. \tag{3.13}$$

Hence

$$\big[[d(x), y], t\big] = 0 \text{ for all } x, y \in U, t \in N. \tag{3.14}$$

Replacing y by $yd(x)$ in (3.14) we find that

$$\big[[d(x), y]d(x), t\big] = 0 \text{ for all } x, y \in U, t \in N. \tag{3.15}$$

In view of (3.13), Eq. (3.15) assures that

$$[d(x), y]N[d(x), y] = \{0\} \text{ for all } x, y \in U. \tag{3.16}$$

By primeness of N, Eq. (3.16) shows that

$$[d(x), y] = 0 \text{ for all } x, y \in U. \tag{3.17}$$

Replacing y by ry for all $r \in N$ in (3.17) and using Lemma 2.2 (ii), we obtain

$$[d(x), r] = 0 \text{ for all } x \in U, r \in N.$$

Hence $d(U) \subseteq Z(N)$ and application of Lemma 2.4, assures that N is a commutative ring.

Theorem 3.4 Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. There is no nonzero derivation d of N such that $d(x)oy = xoy$, for all $x, y \in U$.

Proof Suppose that

$$d(x)oy = xoy \text{ for all } x, y \in U. \tag{3.18}$$

Replacing x by xy in (3.18) we obtain

$$\begin{aligned} d(xy)oy &= xyoy \\ &= (xoy)y \\ &= (d(x)oy)y \\ &= d(x)y^2 + yd(x)y \text{ for all } x, y \in U. \end{aligned}$$

Since $d(xy)oy = d(xy)y + yd(xy)$, according to Lemma 2.1 (i), we obtain

$$d(x)y^2 + xd(y)y + yxd(y) + yd(x)y = d(x)y^2 + yd(x)y \text{ for all } x, y \in U.$$

and therefore

$$yxd(y) = -xd(y)y \text{ for all } x, y \in U. \tag{3.19}$$

Substituting zx for x, where $z \in N$ in (3.19) and using it again, we find that

$$\begin{aligned} yzxd(y) &= -zxd(y)y \\ &= (-z)(xd(y)y) \\ &= (-z)(-yxd(y)) \\ &= (-z)(-y)xd(y) \text{ for all } x, y \in U, z \in N. \end{aligned}$$

The last expression reduced to

$$\big(yz - (-z)(-y)\big)xd(y) = 0 \text{ for all } x, y \in U, z \in N.$$

and therefore

$$\big(-(-y)z + z(-y)\big)Ud(y) = \{0\} \text{ for all } y \in U, z \in N. \tag{3.20}$$

By primeness, (3.20) assures that either $-y \in Z(N)$ or $d(y) = 0$, for all $y \in U$. Suppose there exists $y_o \in U$ such that $d(y_o) = 0$, then by applying our hypothesis, we obtain $y_o\, o\, t = 0$ for all $t \in U$ so $y_o t = -ty_o$ for all $t \in U$. Substituting rt instead of t where $r \in N$, we obtain $[-y_o, r]U = \{0\}$ for all $r \in N$ which implies that $-y_o \in Z(N)$ by Lemma 2.2 (ii). In both cases, we arrive at $-y \in Z(N)$ for all $y \in U$. Replacing y by ny where $n \in N$, we get $-ny \in Z(N)$ for all $y \in U, n \in N$ so $(-n)y \in Z(N)$ for all $y \in U, n \in N$ and using Lemma 2.6 (ii), we conclude that N is a commutative ring.

Therefore (3.18) assures that $d(x)y = xy$ for all $x, y \in U$. Substituting xt for $t \in N$ instead of x, we obtain $d(xt)y = xty$ for all $x, y \in U, t \in N$. So that $d(x)ty = 0$ for all $x, y \in U, t \in N$ i.e., $d(x)Ny = \{0\}$. By primeness of N, we get $d(x) = 0$ for all $x \in U$ or $y = 0$ for all $y \in U$ which gives us a contradiction with Lemma 2.2 (iii) and the fact that $U \neq \{0\}$.

Theorem 3.5 Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. If N admits no nonzero derivation d, then the following assertions are equivalent

(i) $d(x)oy \in Z(N)$ for all $x, y \in U$.

(ii) N is a commutative ring.

Proof It is clear that $(ii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Assume that

$$d(x)oy \in Z(N) \text{ for all } x, y \in U. \tag{3.21}$$

(a) If $Z(N) = \{0\}$, then equation (3.21) reduced to

$$d(x)y = -yd(x) \text{ for all } x, y \in U. \tag{3.22}$$

Substituting zy for y, where $z \in N$ in (3.22), we obtain

$$\begin{aligned} d(x)zy &= -zyd(x) \\ &= (-z)yd(x) \\ &= (-z)(-d(x)y) \\ &= (-z)d(-x)y \quad \text{for all } x, y \in U, z \in N. \end{aligned}$$

In such a way

$$\big(d(x)z + zd(-x)\big)y = 0 \text{ for all } x, y \in U, z \in N.$$

and therefore

$$\big(-d(-x)z + zd(-x)\big)U = \{0\} \text{ for all } x, y \in U, z \in N. \tag{3.23}$$

Since N is prime, then equation (3.23) forces $d(-x) \in Z(N)$ for all $x \in U$ so $d(-U) \subseteq Z(N)$. It is clear that $-U$ is a semigroup right ideal, by Lemma 2.4, we conclude that N is a commutative ring.
(b) Suppose that $Z(N) \neq \{0\}$ and replacing y by $yd(x)$ in (3.21), we get $(d(x)oy)d(x) \in Z(N)$ for all $x, y \in N$. By Lemma 2.6 (ii), we obtain

$$d(x) \in Z(N) \text{ or } d(x)oy = 0 \text{ for all } x, y \in U. \tag{3.24}$$

Suppose there exists $x_o \in U$ such that $d(x_o) \in Z(N)$ and using (3.21), we obtain $d(x_o)(u+u) \in Z(N)$ for all $u \in U$. By Lemma 2.6 (ii), we get $d(x_o) = 0$ or $u + u \in Z(N)$ for all $u \in U$.
Thus (3.24) becomes

$$d(x)\ o\ y = 0 \text{ or } u + u \in Z(N) \text{ for all } x, y, u \in U. \tag{3.25}$$

If $d(x)\ o\ y = 0$ for all $x, y \in U$. Using the same proof of (a), we obtain N is a commutative ring.
If $u + u \in Z(N)$ for all $u \in U$. Replacing u by un, where $n \in N$, we get $(u+u)n \in Z(N)$ for all $u \in U, n \in N$ and using Lemma 2.6 (ii) together with 2-torsion freeness of N, conclude that N is a commutative ring.

Theorem 3.6 Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. Then there exists no nonzero derivation d of N satisfying one of the following conditions:
(i) $d(x\ o\ y) = [x, y]$ for all $x, y \in U$.
(ii) $d([x, y]) = x\ o\ y$ for all $x, y \in U$.

Proof (i) We have

$$d(x\ o\ y) = [x, y] \text{ for all } x, y \in U. \tag{3.26}$$

Suppose $d(x\ o\ y) = [x, y]$ for all $x, y \in U$. Replacing y by yx in (3.26) we arrive at $d((x\ o\ y)x) = [x, y]x$. Hence expanding this relation and using (3.26), we find that

$$(x\ o\ y)d(x) = 0 \text{ for all } x, y \in U. \tag{3.27}$$

Replacing further y by zy for $z \in N$ in (3.27) we find that

$$(x(zy) + (zy)x)d(x) = 0 \text{ for all } x, y \in U, z \in N.$$

Now application of (3.27) yields that $yxd(x) = -xyd(x)$. Combining this fact together with the later relation we arrive at

$$\big(xz + z(-x)\big)yd(x) = 0 \text{ for all } x, y \in U, z \in N.$$

This implies that

$$[-x, z]Ud(x) = \{0\} \text{ for all } x \in U, z \in N. \quad (3.28)$$

By Lemma 2.2 (i), we get $d(x) = 0$ for all $x \in U$ or $-x \in Z(N)$. Both cases give $d(-U) \subseteq Z(N)$ which forces that N is a commutative ring. In this case (3.26) and 2-torsion freeness implies that

$$d(xy) = 0 \text{ for all } x, y \in U. \quad (3.29)$$

This means that

$$d(x)y + xd(y) = 0 \text{ for all } x, y \in U. \quad (3.30)$$

Putting xz instead of x in (3.30) and using (3.29), we arrive at $xUd(y) = \{0\}$ for all $x, y \in U$. Using Lemma 2.2 (i), we get either $x = 0$ or $d(y) = 0$ for all $y \in U$ which gives a contradiction by Lemma (2.2) (iii) and the fact that $U \neq \{0\}$. This complete proof of part (i).

(ii) If N satisfies $d([x, y]) = x \, o \, y$ for all $x, y \in U$, then again using the same arguments we get the required result.

Theorem 3.7 Let N be a 2-torsion free prime near-ring which admits a nonzero derivation d and U be a nonzero semigroup ideal of N such that $U \cap Z(N) \neq \{0\}$. Then the following assertions are equivalent

(i) $d(x \, o \, y) \in Z(N)$ for all $x, y \in U$.

(ii) N is a commutative ring.

Proof It is obvious that $(ii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Suppose that

$$d(x \, o \, y) \in Z(N) \text{ for all } x, y \in U. \quad (3.31)$$

(a) If $Z(N) = \{0\}$, then $d(x \, o \, y) = 0$ and replacing y by yx we obtain $(x \, o \, y)d(x) = 0$ for all $x, y \in U$ and thus

$$xyd(x) = -yxd(x) \text{ for all } x, y \in U. \quad (3.32)$$

Substituting yz where $z \in U$ for y in (3.32), we obtain

$$[-x, y]zd(x) = 0 \text{ for all } x, y, z \in U. \quad (3.33)$$

Accordingly,

$$[-x, y]Ud(x) = \{0\} \text{ for all } x, y \in U. \quad (3.34)$$

Using Lemma 2.2 (i), we get $-x \in Z(N)$ or $d(x) = 0$ for all $x \in U$ which implies that $d(-U) \subseteq Z(N)$. It is clear that $-U$ is a semigroup right ideal, by Lemma 2.4, we conclude that N is a commutative ring.

(b) If $Z(N) \neq \{0\}$, replacing y by yz in (3.31), where $z \in Z(N)$, we get

$$d(x \, o \, y)z + (x \, o \, y)d(z) \text{ for all } x, y \in U, z \in Z(N). \quad (3.35)$$

Using (3.31) together with Lemma 2.2 (ii), (3.35) reduces to

$$(x \, o \, y)d(z) \in Z(N) \text{ for all } x, y \in U, z \in Z(N). \quad (3.36)$$

Since $d(z) \in Z(N)$, (3.36) yields that

$$0 = [(x \, o \, y)d(z), t]$$

$$= [x\ o\ y, t]d(z) \text{ for all } x, y, t \in U, z \in Z(N).$$

So $[x\ o\ y, t]Nd(z) = \{0\}$ for all $x, y, t \in U, z \in Z(N)$. By primeness of N, the last equation forces either

$$d(Z(N)) = \{0\} \text{ or } x\ o\ y \in Z(N) \text{ for all } x, y \in U. \tag{3.37}$$

Suppose that $d(Z(N)) = \{0\}$. If $0 \neq y \in U \cap Z(N)$; since $d(x\ o\ y) = d(x)y + d(x)y \in Z(N)$, then $d(d(x)y + d(x)y) = 0$ and hence

$$(d^2(x) + d^2(x))y = 0 \text{ for all } x, y \in U. \tag{3.38}$$

Using the fact that $0 \neq y \in Z(N)$, Eq. (3.38) leads to $d^2(x) = 0$ for all $x \in U$. So that $d^2 = 0$ and Lemma 2.7 forces $d = 0$, a contradiction.

Accordingly, we have $x\ o\ y \in Z(N)$ for all $x, y \in U$. Let $0 \neq y \in Z(N)$; from $x\ o\ y = (x + x)y$, $x^2\ o\ y = (x^2 + x^2)y$ it follows, because of the primeness, that $x + x \in Z(N)$, $x^2 + x^2 \in Z(N)$ for all $x \in U$. Thus

$$\begin{aligned}(x + x)xt &= (x^2 + x^2)t \\ &= t(x^2 + x^2) \\ &= t(x + x)x \\ &= (x + x)tx \text{ for all } x \in U, t \in N.\end{aligned}$$

and therefore

$$(x + x)N[x, t] = \{0\} \text{ for all } x \in U, t \in N. \tag{3.39}$$

Once again using the primeness hypothesis, Eq. (3.39) yields $x \in Z(N)$ or $2x = 0$ in which case 2-torsion freeness forces $x = 0$. Consequently, in both the cases we arrive at $x \in Z(N)$ for all $x \in U$. Hence $d(U) \subseteq Z(N)$ and Lemma 2.4 assures that N is a commutative ring.

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