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Mildly α*-Normal Spaces and rgα*-Continuous Functions

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Abstract: The aim this paper is to establish and study a new class of spaces, called mildly α*-normal spaces. The relationship with among normal, almost normal, quasi normal, mildly normal, π-normal spaces and their generalizations are investigated. Moreover, we establish rgα*-continuous functions. Utilizing rgα*-continuity, we obtain characterizations and preservation theorems for mildly α*-normal spaces.

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1. Introduction

In 1968, the notion of quasi normal space was introduced by Zaitsev [**15**]. In 1970, Levine [**7**] initiated the study of so called generalized closed (briefly g-closed) sets in order to extend many of the most important properties of closed sets to a large family. In 1968, Singal and Singal **[12]** introduced Almost continuous functions. In 1970, Singal and Arya **[11]** introduced Almost normal and almost completely regular spaces. In 1972, Shchepin **[10]** introduced the notion of mildly normal space and in 1973, Singal and Singal **[13]** independently. In 1990, Lal and Rahman [**6**] have further studied notions of quasi normal and mildly normal spaces. In 2000, Dontchev and Noiri **[1]** introduced the notion of πg-closed sets. By using πg-closed sets, Dontchev and Noiri [**1**] obtained a new characterization of quasi normal spaces. In 2008, Kalantan [**5**] introduced a weaker version of normality called π-normality and proved that π-normality is a property which lies between normality and almost normality. In 2013, Thakur C. K. Raman et al. [**14**] introduced the concepts of α*-generalized and α*-separation axioms in topological spaces. Recently, Jitendra Kumar et al. **[3]** established the concept of rgg*-continuous functions and mildly g*-normal spaces in topological spaces.

2. Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively. A subset A is said to be **regular open** (resp. **regular closed**) if A = int(cl(A)) (resp. A = cl(int(A)). The finite union of regular open sets is said to be **π-open**. The complement of a π-open set is said to be **π-closed**. A is said to be **α*-closed [14]** if α-cl(A) ⊂ U whenever A ⊂ U and U is αg-open in X. The complement of a α*-closed set is said to be α*-open. The intersection of all α*-closed sets containing A is called **α*-closure** of A, and is denoted by **α*-cl(A) [2]**. The **α*-interior** of A, denoted by **α*-int(A) [2]**, is defined as union of all α*-open sets contained in A.

2.1 Definition. A subset A of a space (X, τ) is said to be
(1) generalized closed (briefly **g-closed**) [**7**]) if cl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(2) **regular generalized closed** (briefly **rg-closed [9]**) if cl(A) ⊂ U whenever A ⊂ U and U is regular open in X.
(3) **α*-closed [14]** if α-cl(A) ⊂ U whenever A ⊂ U and U is αg-open in X.
(4) **generalized α*-closed** (briefly **gα*-closed [2]**)) if α*-cl(A) ⊂ U whenever A ⊂ U and U is α*g- open in X.
(5) **regular generalized α*-closed** (briefly **rgα*-closed**) if α*cl(A) ⊂ U whenever A ⊂ U and U is regular open in X.
(6) **g-open** (resp**. rg-open**, **gα*-open, rgα*-open**) if the complement of A is g-closed (resp. rg-closed, gα*-closed, rgα*-closed)

2.2 Remark. We have the following implications for the properties of subsets:

closed	⇒	g-closed	⇒	rg-closed
⇓		⇓		⇓
α*-closed	⇒	gα*-closed	⇒	rgα*-closed

Where none of the implications is reversible as can be seen from the following examples:

2.3 Example. Let X = {a, b, c} and τ = {ϕ, {a}, X}. Then A = {b} is g-closed but it is not closed.

2.4 Example. Let X = {a, b, c} and ℑ = {ϕ, {a}, X}. Then A = {b} is g-closed as well as αg-closed. Hence A is gα*-closed. But it is not closed.

2.5 Example. Let X = {a, b, c, d} and ℑ = {ϕ, {c}, {d}, {a, c}, {c, d}, {a, c, d}, X}. Then A = {a} is α-closed as well as αg-closed. Hence A is gα*-closed. But it is not closed.

2.6 Example. Let X = {a, b, c} and ℑ = {ϕ, {a}, {b}, {a, b}, X}. Then A = {a, b} is rg-closed as well as rαg-closed. Hence A is rgα*-closed. But it is not αg- closed.

2.7 Example. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, X}. Then A = {a, b} is g-closed as well as αg-closed. But it is not α-closed.

2.8 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d} X}. Then A = {a, b} is rαg-closed as well as rgα*- closed. But it is not closed.

2.9 Lemma

A subset A of a space X is rgα*-open if and only if F ⊂ α*-int(A) whenever F is a regular closed and F ⊂ A.

3. Mildly α*-normal Spaces

3.1 Definition

A topological space X is said to be **quasi α*-normal [2]** (resp. **quasi-normal [15]**) if for every pair of disjoint π-closed subsets H, K, there exist disjoint α*-open (resp. open, p-open) sets U, V of X such that H ⊂ U and K ⊂ V.

3.2 Definition

A topological space X is said to be **mildly α*-normal** (resp. **mildly-normal [7, 10]**) if for every pair of disjoint regular closed subsets H, K, there exist disjoint α*-open (resp. open, p-open) sets U, V of X such that H ⊂ U and K ⊂ V.

3.3 Definition

A topological space X is said to be **almost α*-normal [4]** (resp. **almost normal [11]**) if for any two disjoint closed subsets A and B of X, one of which is regular closed, there exist disjoint α*-open (resp. open, p-open) sets U, V of X such that A ⊂ U and B ⊂ V.

3.4 Example Let X = {a, b, c} and τ = {ϕ, {a}, {a, b}, {a, c}, X}. Then (X. τ) is almostα*-normal space, but it is not α*-normal, since the pair of disjoint closed sets {b} and {c} have no disjoint α*-open sets containing them.

3.5 Example. Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, c}, {a, b, d}, X}. Then X is rg-normal.

3.6 Example. Let X = {a, b, c, d} and τ = {ϕ, {a}, {c}, {a, c}, { b, d}, {a, b, d}, {b, c, d}, X}. The pair of disjoint π-closed subsets of X are A = {a} and B = {c}. Also U = {a} and V = {b, c, d} are disjoint open sets such that A ⊂ U and B ⊂ V. Hence X is quasi-normal as well as quasi α*-normal because every open set is α*-open set.

3.7 Example. Let X = {a, b, c} and τ = {ϕ, {a}, {a, b}, {a, c}, X}. (X, τ) is almost normal space, but it is not normal.

3.8 Example. Let X = {a, b, c, d} and τ = {ϕ, {c}, {d}, {a, b}, {c, d}, {a, b, c}, {a, b, d}, X}. The pair of disjoint regularly closed subsets of X are A = {c} and B = {d}. Also U = {b, c} and V = {a, d} are α*-open sets such that $A \subset U$ and $B \subset V$. Hence X is mildly α*-normal. But the space X is neither mildly normal nor normal, since U and V are not open sets.

By the definitions and examples stated above, we have the following diagram:

normal	⇒	π-normal	⇒	almost-normal	⇒	mildly-normal
⇓		⇓		⇓		⇓
α*-normal	⇒	πα*-normal	⇒	almost α*-normal	⇒	mildly α*-normal

and

normal	⇒	π-normal	⇒	quasi-normal	⇒	mildly-normal
⇓		⇓		⇓		⇓
α*-normal	⇒	πα*-normal	⇒	quasi α*-normal	⇒	mildly α*-normal

3.9 Theorem.

The following are equivalent for a space X:

1) X is mildly α*-normal.
2) For every pair of regularly open sets U and V of X whose union is X, there exist α*-closed sets A, B such that $A \subset U$, $B \subset V$ and $A \cup B = X$.
3) For every regularly closed set F and every regularly open set G containing F, there exists a α*-open set U such that $F \subset U \subset \alpha^*\text{-cl}(U) \subset G$.

Proof: (1) ⇒ (2). Let U and V be a pair of regularly open sets in a mildly α*-normal space X such that $X = U \cup V$. Then $X - U$, $X - V$ are disjoint regularly closed sets. Since X is mildly α*-normal, there exist disjoint α*-open sets U_1 and V_1 such that $X - U \subset U_1$ and $X - V \subset V_1$. Let $A = X - U_1$, $B = X - V_1$. Then A, B are α*-closed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.
(2) ⇒ (3). Let be a regularly closed set and G be a regularly open set containing F. Then $X - F$ and G are regularly open sets whose union is X. Then by (2), there exist α*-closed sets W_1 and W_2 such that $W_1 \subset X - F$ and $W_2 \subset G$ and $W_1 \cup W_2 = X$. Then $F \subset X - W_1$, $X - G \subset X - W_2$ and $(X - W_1) \cap (X - W_2) = \phi$. Let $U = X - W_1$ and $V = X - W_2$. Then U and V are disjoint α*-open sets such that $F \subset U \subset X - V \subset G$. As $X - V$ is α*-closed set, we have $\alpha^*\text{-cl}(U) \subset X - V$ and $F \subset U \subset \alpha^*\text{-cl}(U) \subset G$.
(3) ⇒ (1). Let F_1 and F_2 be two disjoint regularly closed sets of X. Put $G = X - F_1$, then $F_1 \cap G = \phi$. $F_1 \subset G$, where G is a regularly open set. Then by (3), there exists a α*-open set U of X such that $F_1 \subset U \subset \alpha^*\text{-cl}(U) \subset G$. It follows that $F_2 \subset X - \alpha^*\text{-cl}(U) = V$, says, then V is α*-open and $U \cap V = \phi$. Hence F_1 and F_2 are separated by α*-open sets U and V. Therefore X is mildly α*-normal.

3.10 Theorem.

The following are equivalent for a space X:

1) X is mildly α*-normal.
2) For any disjoint regular closed sets A and B, there exist disjoint gα*-open sets U and V such that $A \subset U$ and $B \subset V$.
3) For disjoint regular closed sets A and B of X, there exist disjoint rgα*-open sets U and V such that $A \subset U$ and $B \subset V$.
4) For any regular closed set A and any regular open set V containing A, there exists a gα*-open set U of X such that $H \subset U \subset \alpha$-*cl(U) $\subset V$.
5) For any regular closed set A and any regular open set V containing A, there exists a rgα*-open set U of X such that $A \subset U \subset \alpha$-*cl(U) $\subset V$.

Proof:

(1)$\Rightarrow$(2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1).

(1)$\Rightarrow$(2). Let X be mildly α*-normal space. Let A, B be disjoint regular closed sets of X. By assumption, there exist disjoint α*-open sets U, V such that $A \subset U$ and $B \subset V$. Since every α*-open set is gα*-open, so, U and V are gα*-open sets such that $H \subset U$ and $K \subset V$.

(2)$\Rightarrow$(3). Let A, B be two disjoint regular closed sets. By assumption, there exist gα*-open sets U and V such that $A \subset U$ and $B \subset V$. Since gα*-open set is rgα*-open, so, U and V are rgα*-open such that $H \subset U$ and $K \subset V$.

(3)$\Rightarrow$(4). Let A be any regular closed set and V be any regular open set containing A. By assumption, there exist rgα*-open sets U and W such that $A \subset U$ and $X - V \subset W$. By **Lemma 2.9,** we get $X - V \subset \alpha$*-int(W) and $U \cap \alpha$*-int(W) = ϕ. Therefore, we obtain α*-cl(U) $\cap$ α*-int(W) = ϕ and hence $A \subset U \subset \alpha$*-cl(U) $\subset X - \alpha$*-int(W) $\subset V$.

(4)$\Rightarrow$(5). Let A be any regular closed set and V be any regular open set containing A. By assumption, there exist gα*-open set U of X such that $H \subset U \subset \alpha$*-cl(U) $\subset V$. Since, every gα*-open set is rgα*-open, there exist rgα*-open sets U of X such that $H \subset U \subset \alpha$*-cl(U) $\subset V$.

(5)$\Rightarrow$(1). Let A, B be any two disjoint regular closed sets of X. Then $H \subset X - K$ and $X - B$ is regular open. By assumption, there exists rgα*-open set G of X such that $A \subset G \subset \alpha$*-cl(G) $\subset X - B$. Put U = α*-int(G), V = K – α*-cl(G). Then U and V are disjoint α*-open sets of X such that $A \subset U$ and $B \subset V$.

Using **Theorem 3.10,** it is easy to produce the following theorem, which is a Urysohns Lemma version for mildly α*-normal. A proof can be introduced by a similar way of the normal case.

3.11 Theorem

A space X is mildly α*-normal iff for every pair of disjoint regularly closed sets A and B of X, there exists a continuous function f on X into [0, 1], with its usual topology, such that f(A) = {0} and f(B) = {1}.
It is easy to see that the inverse image of a regularly closed set under an open continuous function is regularly closed. It will use in the next theorem.

3.12 Theorem

Let X is a mildly α*-normal space and f : X → Y is an open continuous injective function. Then f(X) is a softly normal space.

Proof. Let A and B be any two regularly closed subset of f(X) such that A ∩ B = ϕ. Then $f^{-1}(A)$ and $f^{-1}(B)$ regularly closed sets of X. Since X is mildly α*-normal, there are two disjoint α*-open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Since f is one-one and open, since every open set is α*-open set, result follows.

3.13 Corollary

Mild α*-normality is a topological property.

3.14 Definition. A function f : X → Y is said to be

1) **rgα*-continuous** if $f^{-1}(F)$ is rgα*-closed in X for every closed set F of Y.
2) **α*-rgα*-continuous** if $f^{-1}(F)$ is rgα*-closed in X for every α*-closed set F of Y.
3) **rgα*-irresolute** if $f^{-1}(F)$ is rgα*-closed in X for every rgα*-closed set F of Y.
4) **rc-preserving [8]** (resp. **almost closed [12]**) if f(F) is regular closed (resp. closed) in Y for every regular closed set F of X.

3.15 Theorem

If f : X → Y is a α*-rgα*-continuous, rc-preserving injection and Y is mildly α*-normal then X is mildly α*-normal.

Proof. Consider A and B be any disjoint regular closed sets of X. Since f is an rc-preserving injection, f(A) and f(B) are disjoint regular closed sets of Y. By mild α*-normality of Y, there exist disjoint α*-open sets U and V of Y such that $f(A) \subset U$ and $f(B) \subset V$. Since f is α*-rgα*-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint rgα*-open sets containing A and B respectively. Hence by **Theorem 3.10,** X is mildly α*-normal.

3.16 Theorem

If f : X → Y is a α*-rgα* continuous, almost closed surjection and Y is α*-normal space, then X is mildly α*-normal.

Proof. Similar to preceding one.

4. Conclusion

In this paper, we established a new class of spaces, called mildly α*-normal spaces and introduced their relationships with some weak forms of normal spaces like normal, almost normal, quasi normal, mildly normal, π-normal spaces and their generalizations in topological spaces.

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