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CONTRIBUTIONS OF SOME EUROPEAN MATHEMATICIANS IN THE DEVELOPMENT OF PROBABILITY THEORY: A HISTORICAL SEARCH

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ABSTRACT

In the historical search of contributions of some European Mathematicians in the development of Probability Theory, we have given particular emphasis on the works of Abraham De Moivre (1667-1754.), Thomas Bayes (1701-1761), Leonhard Euler (1707-1783), Pierre S. Laplace (1749-1827), Johann Carl Friedrich Gauss (1777-1855) and Sime/on Denis Poisson (1781-1840). The aim of this article is to highlight the contributions made by the well-known European Mathematicians towards development of Probability Theory.

Key Words: Probability Theory, Dirac delta function, Beta function, Gamma function.

1. Introduction:

R. A. Fisher [1959], once mentioned '*More attention to the History of Science is needed, as much by scientists as by historians, . . . and this should mean a deliberate attempt to understand the thoughts of the great masters of the past, to see in what circumstances or intellectual milieu their ideas were framed, where they took the wrong turning or stopped short on the right track. A sense of the continuity and progressive and cumulative character of an advancing science is the best prophylactic I can suggest against the manic-depressive alternations of the cult of vogue*'.

Concepts of Probability have been around for thousands of years, but the Probability Theory did not arise as a part of Mathematics until the mid-seventeenth century. During 15th century, calculations of Probabilities became more noticeable. In1494, Fra Luca Paccioli (1447-1517) wrote the first printed work on Probability, '*Summa de arithmetica, geometria, proportioni e proportionalita*'. In 1550, Geronimo Cardano (1501-1576), inspired by the *Summa* wrote a book about games of chance entitled '*Liber de Ludo Aleae*' which was published after 100 years of his death [1]. Cardano presented the result of his theory on dice and with respect to the roll of a die. He wrote:

"*...in six casts each point should turn up once; but since some will be repeated, it follows that others will not turn up*."

The use of '*should turn up*' [1] in the articulation of this principle suggests that it is based upon the symmetry of the die having six sides, each of which is as likely as the other to occur. It is an intuitive concept and is used to introduce elementary probability calculation, even today.

Galileo Galilei (1564 -1642), published an article giving explanation of observations from a random process. Galileo wrote '*Sopra Le Scopertedei Dadi*' in response to a request for an explanation about an observation concerning the playing of three dice where possible combinations of dice sides totalling 9, 10, 11, and 12 are the same. In Galileo's word [1]:

"*…it is known that long observation has made dice-players consider 10 and 11 to be more advantageous than 9 and 12*."

Galileo explained the phenomenon by enumerating the possible combinations of the three numbers composing the sum. He was able to show that 10 will show up in 27 ways out of all possible throws (which Galileo indicated as 216). Since 9 can be found in 25 ways and this explains why it is at a 'disadvantage' in comparison to 10 [1].

In the mid-seventeenth century, a simple question was directed to Blaise Pascal (1623-1662) by a nobleman Chevalier de Méré. It is believed that this question led to the birth of Probability Theory. Chevalier de Méré betted on a roll of a die that at least one 6 would appear during a total of four rolls. From past experience, he knew that he was more successful than not with this game of chance. Tired of his approach, he decided to change the game. He then changed the bet to that he would get a total of 12, or a double 6, on twenty-four rolls of two die. Later, he realized that his old approach to the game resulted in more money. So, he asked his friend Blaise Pascal why his new approach was not profitable. Pascal worked through the problem and found that the probability of winning using the new approach was only 49.1% compared to 51.8% using the old approach.

Christian Huygens's(1629 -1695) book entitled '*On Reasoning in Games of Chance*' was cited in the literature as the first published mathematical work on Probability and in 1657 it was first printed. Huygen mentioned [1]: "*Although games depend entirely upon Fortune and success is always uncertain; yet it may be exactly determined at the same time how much more probability there is that [one] should lose than win*"

In 1710, Dr. John Arbuthnott (1667 – 1735), a friend of Jonathan Swift (1667 – 1745) and Isaac Newton (1643 – 1727), wrote a paper entitled '*An Argument for Divine Providence, taken from the constant Regularity observed in the Births of both Sexes*'. This paper is very important because it is one of the first applications of Probability to phenomena other than games of chance [1]. Dr. Arbuthnott suggested in his paper: *"I believe the Calculation of the Quantity of Probability might be improved to a very useful and pleasant Speculation, and applied to a great many events which are accidental, besides those of Games*."

In 1713 Jacob Bernoulli(1655 –1705), a mathematician wrote the book on Probability entitled '*Arts Conjectandi*', which discussed a number of problems requiring Probability and considered the nature of Probability[1]. In the book he wrote: "*…probability is a degree of certainty and differs from absolute certainty as a part differs from the whole. If, for example, the whole and absolute certainty – which we designate by the letter a or by the unity symbol 1 – is supposed to consist of five probabilities or parts, three of which stand for the existence or future existence of some event, the remaining two standing against its existence or future existence, this event is said to have 3/5a or 3/5 certainty*."

Bernoulli also mentioned that probability is a consequence of uncertainty: "*…those data which are supposed to determine later events (and especially such data which are in nature) have nevertheless not been learned well enough by us*."

In Chapter IV of Part IV of "Arts Conjectandi", Bernoulli wrote: *"Something further must be contemplated here which perhaps no one has thought of about till now. It certainly remains to be inquired whether after the number of observations has been increased, the probability is increased of attaining the true ratio between the numbers of cases in which some event can happen and in which it cannot happen, so that the probability finally exceeds any given degree of certainty…*".The proposed solution to this inquiry, is called today the Law of Large Numbers [1].

In 1975, Ian Hacking in his book [1] '*The Emergence Of Probability*' mentioned : "*I am inclined to think that the preconditions for the emergence of our concept of probability determine the very nature of this intellectual object probability…..the preconditions for the emergence of probability determined the space of possible theories about probability. That means that they determine, in part, the space of possible interpretations of quantum mechanics, of statistical inference, and of inductive logic.*"

Hacking mentioned about evidences which suggest that Probability concepts were found earlier in the Eastern world, i.e. in India and Arab.

In the next sections of this paper, an attempt has been made to describe some significant contributions in the field of Probability Theory by some well-known European Mathematicians.

2. Contribution to Probability by Abraham De Moivre:

2.1 Abraham De Moivre's personal life:

Abraham De Moivre was born on 26th May 1667, at Vitry-le-Francois, France. In 1684 Moivre started to study Mathematics at Paris. The persecution of the French Protestants caused him to seek asylum in England at the age of 21. For the rest of his life he lived in London. He started his livelihood as a private tutor of Mathematics and later also as a consultant to gamblers and insurance brokers. In 1697, he became a prominent Mathematician and a Fellow of the Royal Society. He wrote three outstanding books: *Miscellanea Analytica* (1730) containing papers on mathematics and probability theory, *The Doctrine of Chances*: *A Method of Calculating the Probability of Events in Play* (1718, with reprints in 1738, 1756), and *Annuities upon Lives* (1725, with reprints in 1743, 1750, 1752). Each new edition of the books was an enlarged version of the previous one. His second book contained solutions to old problems and an astounding number of new problems with solutions. He was also known for his '*de Moivre's formula'*. Further, he made contributions towards the development of Normal distribution. According to available literature, it is known that he was highly influenced by his friend Isaac Newton. He died on 27th November 1754 at London, UK.

2.2. Contribution to Probability Theory by Abraham De Moivre:

In 1733, De Moivre derived a formula for approximating binomial sums or probabilities when the number of trials is sufficiently large converges to normal probability distribution, which is later popularly known as Central Limit Theorem.

William Adams, in his book entitled '*The Life and Times of the Central Limit Theorem'*, stated: *"De Moivre did not name* $\frac{\sqrt{n}}{2}$*, which is what we would today call standard deviation within the context considered, but … he referred to* $\sqrt{n}$ *as the Modulus by which we are to regulate our estimation*."

2.2.1. De Moivre's Central Limit Theorem:

In 1733 at the age of 66, De Moivre's made one of his most significant contributions through modification of the Central limit theorem. He considered the form of his Central limit theorem as a generalization of Bernoulli's main theorem given in the book *Ars Conjectandi*, written by Bernoulli. This was later named as Law of Large Numbers by Poisson [3].

According to De Moivre: "*The random phenomena that obey the normal probability law can be obtained by examining the manner in which the normal density function and normal distribution function first arose in the probability theory as means of approximately evaluating probabilities associated with the binomial probability law*".

Central Limit Theorem:

The probability that a random phenomenon obeying the binomial probability law with parameters n and p will have an observed value lying between a and b (inclusive), is given approximately by

$$\sum_{k=a}^{b} {}^{n}c_k\, p^k (1-p)^{n-k} = \frac{1}{\sqrt{2\pi}} \int_{\frac{a-np-\frac{1}{2}}{\sqrt{np(1-p)}}}^{\frac{b-np+\frac{1}{2}}{\sqrt{np(1-p)}}} e^{\frac{-z^2}{2}}\, dz$$

This particular Central Limit Theorem (CLT) was stated by De Moivre in 1733 for the case $p = \frac{1}{2}$ and established for arbitrary value of p by Laplace in 1817. Hence in some literature, the above CLT is also stated as De Moivre-Laplace theorem. Here we do not discuss the proof of the theorem as it is available in the standard text books. However for an elementary rigorous proof of the said theorem, the interested reader should consult J. Neyman (*First course in Probability and Statistics*, New Work, Henry Holt, 1950, pages 234-242.)

De Moivre based his work on the results achieved by Jacob and Nicolaus Bernoulli concerning binomial coefficients and their sums. He stated for the simplest case, i.e. the symmetric binomial $p = \frac{1}{2}$ and $\varepsilon = \frac{1}{2}$. De Moivre first estimated the maximum term $b(m)$ in the sum as: $\sum_{k=m-l}^{m+l} {}^{n}C_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = 2^{-n} \sum_{k=m-l}^{m+l} {}^{n}C_k = \sum_{i=-l}^{l} b(m+i)$ and then for large $n(n = 2m)$ as $b(m) \cong \frac{2}{\sqrt{2\pi n}}$

Subsequently, de Moivre considered the ratio of a term $b(m \pm l)$ with a distance l $\left(l = o\left(\sqrt{n}\right)\right)$ from the maximum to the maximum term for which de Moivre found that for large n:

$$\ln \frac{b\,(m \pm l)}{b\,(m)} \cong \frac{-2l^2}{n}$$

Between 1721 and 1733 de Moivre worked on these estimations. For the final form of these estimations, asymptotic series for $n!$, where n is large, was important and both de Moivre and Stirling contributed towards it and they published it in 1730. For the development of this asymptotic series de Moivre used Jacob Bernoulli's formulae for sums of powers of integers extensively.

3. Contribution to Probability by Thomas Bayes:

3.1. Thomas Bayes's personal life:

Thomas Bayes was an English Statistician and Philosopher. Bayes was born in 1701 in London, England. He was famous for his work 'Bayes' Theorem'. He was the son of London Presbyterian minister Joshua Bayes. In 1719 Thomas Bayes enrolled at the University of Edinburg to study logic and theology. He is known for his two famous publications, the first in 1731, entitled '*Divine Benevolence*' and the second in 1736, entitled '*An Introduction to the Doctrine of Fluxions and a Defence of the Mathematicians Against the Objections of the Author of The Analyst*'. In 1742 Bayes was elected as a Fellow of the Royal Society. In his later years he took a deep interest in Probability Theory. His works and findings on Probability Theory were passed in manuscript form to his friend Richard Price after his death. In1755 he became ill and died in 1761 at Tunbridge Wells.

3.2Contribution to Probability Theory by Thomas Bayes:

3.2.1. Bayes' rule or theorem:

Thomas Bayes made some important fundamental contributions to Probability Theory which is based on the notion of conditional probability.

Fundamental idea of 'conditional probability' denoted by $P(A/B)$ is that probability of an event A, given that B has already occurred, is given by

$$P(A/B) = \frac{P(AB)}{P(B)}\,,\ P(B) \neq 0 \qquad (1)$$

Or, equivalently,

$$P(AB) = P(B)P(A/B) \qquad (2)$$

This result (2) is known as the multiplication law for determining the probability when each of the two events, A and B, occurs. However, the use of this multiplication law is not the only way to determine the probability that each of two events occurs. Since $P(AB) = P(BA)$, therefore it follows that:

$$P(BA) = P(B/A)\, P(A) = P(A/B)P(B) \qquad (3)$$

Dividing (3) by $P(B)$ leads to the celebrated Bayes' theorem:

$$P(A/B)=\frac{P(B/A)P(A)}{P(B)}\ ,P(B)\neq 0 \qquad (4)$$

Equation (4) gives the 'conditional' or 'inverse' probability of *A* given *B* in term of the conditional probability of *B* given *A*. In fact, it also provides a direct relation between a posteriori probability, $P(A/B)$ and a priori probability, *P* (*A*) through the likelihood function $P(B/A)$[5]. In other words, a priori probability *P* (*A*) is combined with likelihood function to determine a posteriori probability $P(A/B)$. According to Bayes, it is possible to assign a probability to an event before a similar event has occurred. This is exactly opposite to which asserts that probability of a future event depends on its occurrence in the past. But it is not always clear how to determine a priori probability *P*(*A*). Classical probabilities dealing with throwing of a coin or dice, or drawing from a deck of cards or similar problems are in effect Bayesians where symmetry or subjective intuitive considerations helped to assign probabilities to hypothetical events. However, it is an interesting and subtle problem to develop methods for determining a priori probability. One classical method, first introduced by Jacob Bernoulli and then advocated by Laplace is called the Principle of Insufficient Reason (PIR) which is a rule for assigning probability of *m* events in $n(\geq m)$ mutually exclusive and indistinguishable events so that the probability is $\frac{m}{n}$.

3.2.2. Extended form of Bayes' rule:

Let $A_1, A_2, \ldots A_n$ be *n* events, each of positive probability, which are mutually exclusive and are also exhaustive i.e. the union of all events $A_1, A_2, \ldots A_n$ is equal to certain event. Then, for any subsequent event *B*, one may express probability of *B* in terms of the conditional probabilities $P(B/A_1),\ P(B/A_2),\ \ldots P(B/A_n)$ and unconditional probabilities $P(A_1), P(A_2),\ \ldots P(A_n)$ as follows:

$$P(B)=P(B/A_1)P(A_1)+P(B/A_2)P(A_2)+\ldots+P(B/A_n)P(A_n) \qquad (5)$$

If $A_1 \cup A_2 \cup \ldots \cup A_n = S$, and $A_i \cap A_j = \phi$ for $i \neq j$.

Here *S* denotes sample space and $P(A_i) > 0$

Then equation (5) follows immediately from the relation

$$P(B)=P(BA_1)+P(BA_2)+\ldots+P(BA_n) \qquad (6)$$

due to the fact that $P(BA_i)=P(B/A_i)P(A_i)$ for any event A_i.

There is an interesting consequence to (5), which has led to much philosophical speculation and has been the source of much controversy. Let $A_1, A_2, \ldots A_n$ be '*n*' mutually exclusive and exhaustive events and '*B*' be an event for which one knows the conditional probabilities $P(B/A_i)$ of *B*, for given A_i and also the absolute probabilities $P(A_i)$. One may then compute the conditional probabilities $P(A_i/B)$ of any one of the event A_i for given *B* by following formula

$$P(A_i/B)=\frac{P(BA_i)}{P(B)}=\frac{P(B/A_i)P(A_i)}{\sum_{i=1}^{n}P(B/A_i)P(A_i)} \qquad (7)$$

The above relation (7) is called Bayes's Theorem or Bayes's Formula which was published in a paper in Biometrics, vol. 46(1958) p.p. 293-315.

If the event A_i called 'cause', then Bayes's formula can be regarded as a formula for probability that the event *B*, which has occurred, is the result of the 'cause' A_i. In this way (7) has been interpreted as a formula for the probabilities of 'cause' or 'hypotheses'. The use of Bayes's formula to evaluate probabilities during the course of play of a bridge game is illustrated by Dan F. Waugh and Frederick V. Waugh, in '*On Probabilities in Bridge*', which was published in the journal of the America Statistical Association, vol.48(1953), p.p. 79-87.

4. Contribution to Probability by Euler:

4.1. Euler's personal life:

Leonhard Euler was a famous Swiss Mathematician, Physicist, Astronomer and Engineer. He was born on 15th April 1707, in Basel, Switzerland. His father Paul III Euler, was a pastor of the Reformed Church and mother Marguerite née Brucker, was another pastor's daughter. Leonhard Euler had two younger sisters, Anna Maria and Maria Magdalena and had a younger brother, Johann Heinrich. After the birth of Leonhard Euler, his father moved from Basel to the town of Riehen in Switzerland. Leonhard spent most of his childhood there. Leonhard Euler's father was a friend of great European mathematician Johann Bernoulli and Johann Bernoulli influenced young Leonhard towards mathematics.

Leonhard Euler's formal education started in Basel, where he was sent to live with his maternal grandmother. In 1720, at the age of thirteen, he enrolled at the University of Basel. In 1723, he received a Master of Philosophy with a dissertation that compared the philosophies of Descartes and Newton. As per his father's wish, Euler studied the subjects Theology, Greek and Hebrew. But Bernoulli convinced his father that Leonhard was destined to become a great mathematician. In 1726, Euler completed a dissertation on the propagation of sound with the title '*De Sono'*. At that time Johann Bernoulli's two sons, Daniel and Nicolaus were working at the Imperial Russian Academy of Sciences in Saint Petersburg. On 31st July 1726, Nicolaus died of appendicitis after spending less than a year in Russia. When Daniel assumed his brother's position in the mathematics/physics division, he recommended that the post in physiology that he had vacated be filled by his friend Euler. In November 1726 Euler eagerly accepted the offer.

Leonhard Euler arrived in Saint Petersburg on 17th May 1727. He was promoted from his junior post in the medical department of the academy to a senior position in the mathematics department. He lodged with Daniel Bernoulli with whom he often worked in close collaboration. In 1731, Euler was made a professor of physics. Two years after, Daniel Bernoulli was fed up with the censorship and hostility at Saint Petersburg and he left for Basel. Euler succeeded him as the head of the mathematics department.

On 7 January 1734, he married Katharina Gsell (1707–1773), a daughter of Georg Gsell, a painter from the Academy Gymnasium. The young couple bought a house by the Neva River. Of their thirteen children, only five survived childhood. A fire in St. Petersburg in 1771 cost him his home and almost his life. In 1773, he lost his wife Katharina after 40 years of marriage. Three years after his wife's death, Euler married her half-sister, Salome Abigail Gsell (1723–1794). This marriage lasted until his death. In 1782 he was elected a Foreign Honorary Member of the American Academy of Arts and Sciences.

In St. Petersburg on 18 September 1783, after a lunch with his family, Euler was discussing about the newly discovered planet Uranus and its orbit with a fellow academician Anders Johan Lexell, when he collapsed due to a brain hemorrhage. He died a few hours later. Euler was buried next to Katharina at the Smolensk Lutheran Cemetery on Goloday Island.

He made important and influential discoveries in infinitesimal calculus, graph theory, topology and analytic number theory. He is also known for his work in mechanics, fluid dynamics, optics , astronomy and music theory.

Louis Gustava du Pasquier wrote : "*Euler devoted a portion of his universal interest to the study of the theory of risk and ... to questions involving the calculus of probability*."

4. 2.Contribution to Probability Theory by Euler:

In 1749, an Italian businessman named Roccolini approached Frederick the Great (King of Prussia) with a proposal to establish a lottery system involving the drawing of five numbers from 1 to 90. The King sent the proposal to his scientific advisor Euler for a mathematical review concerning the implementation of a state lottery in Germany. Euler became very interested in analyzing the various aspects of the Genoese Lottery system. He came up with an improved lottery system after addressing combinatorial issues in the analysis of this game of chance.

In addition to his study and research on the royal assignment concerning mathematical problems of lotteries, Euler prepared seven notebooks during his early years in Basel, his first period of stay in St. Petersburg period and then in Berlin. These notebooks contained solutions of many mathematical problems concerning probability theory and statistics, combinatorics, mortality, the mathematical theory of games, commercial as well as life insurance. Under his direct supervision, his research assistant, N. I. Fuss published a textbook in 1776 covering basic elements of probability and statistics, insurance and organization of lotteries with tables. All of his work clearly provided a clear evidence of his interest in Probability Theory and Mathematical Statistics.

Euler wrote four mathematical papers on the probability of calculus based on solutions of various difficult questions in the Genoese Lottery. In 1763, he made a presentation on Reflections on a singular type of lottery called the Genoese Lottery and in 1862, it was published in Euler's '*Opera PosthumaI'*[11].

In this work, Euler introduced the Hypergeometric distribution of the probability $p_{k,m}$ with parameters $l, n \; and \; k$ as

$$p_{k,m} = \frac{\binom{n}{m}\binom{l-n}{k-m}}{\binom{l}{k}}$$

where a player bets on k numbers to match m of them , l and n are the parameters representing n tokens at random from a set numbered 1, 2, 3,· · · , l. Euler also gave a complete derivation of the desired probabilities for four problems with k = 1, 2, 3, 4 and m = 0, 1, 2, 3, and 4.

In 1751, Leonard Euler mentioned in '*M´emoires de l'acad´emie de Berlin*' [8, 11] as:

'*if an ordered set of objects is randomly permuted, what is the probability that none of the objects returns to the original position*?'

In other words, the major problem was to find the probability p_n of derangements $\prod(n)$ so that none of the objects is being returned to its original position. According to Euler, the probability p_n is given by

$p_n = \frac{\prod(n)}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}$ where $\prod(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$, Euler introduced the notation $\prod(n)$ to represent the number of permutations of the n letters $a, b, c, d, e, \cdots$, in which none occupies its original position. Such a permutation is now known as a *derangement*.

It is important to find p_n for small and large values of n. In 1751, Euler calculated the limit of p_n as $n \to \infty$ as

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right) = \frac{1}{e} \approx 0.36787944\cdots$$

where e is the universal exponential constant.

Example:

If the number of cards is equal to *n*, then, expectation of A to win

On the first move = $\frac{1}{n}$

On the second move $= \frac{1}{n} - \frac{1}{n(n-1)}$

On the third move $= \frac{1}{n} - \frac{2}{n(n-1)} + \frac{1}{n(n-1)(n-2)}$

On the fourth move $= \frac{1}{n} - \frac{3}{n(n-1)} + \frac{3}{n(n-1)(n-2)} - \frac{1}{n(n-1)(n-2)(n-3)}$

On the fifth move $= \frac{1}{n} - \frac{4}{n(n-1)} + \frac{6}{n(n-1)(n-2)} - \frac{4}{n(n-1)(n-2)(n-3)} + \frac{1}{n(n-1)...(n-4)}$

On the sixth move $= \frac{1}{n} - \frac{5}{n(n-1)} + \frac{10}{n(n-1)(n-2)} - \frac{10}{n(n-1)...(n-3)} + \frac{5}{n...(n-4)} - \frac{1}{n...(n-5)}$

Therefore, the expectation of A to win in general, at any move will be expressed by the sum of all these 1

Now, the number of these formulae being equal to the number of cards i.e. *n*,

The sum of all the first terms = $\frac{1}{n} + \frac{1}{n} + \cdots \frac{1}{n}\ (n\ times) = n \cdot \frac{1}{n} = 1$

The sum of the numerators of all second terms = $1 + 2 + ... + n - 1 = \frac{n(n-1)}{1.2}$

The sum of all the second terms will be = $\frac{1}{1.2}$

The sum of the numerators of all third terms $1 + 3 + 6 + 10 + ... + \frac{(n-1)(n-2)}{1.2} = \frac{n(n-1)(n-2)(n-3)}{1.2.3}$

The sum of the third terms is = $\frac{1}{1.2.3}$

The sum of the fourth = $\frac{1}{1.2.3.4}$

The sum the fifth = $\frac{1}{1.2.3.4.5}$

and so on.

Hence, in tabular form, the above results can be written as:

Number of cards	Expectation of A to win
1	1
2	$1 - \frac{1}{1.2}$
3	$1 - \frac{1}{1.2} + \frac{1}{1.2.3}$
4	$1 - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4}$
5	$1 - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5}$

From the pattern, it is clear that in any term of the sequence, there will be as many terms as the number of cards. Since the number of total cards is equal to *n*, so, expectation of A not to win will be

$$1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) = \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right)$$

Thus, we have $p_n = \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right)$

For $n \to \infty$, $\lim_{n\to\infty} p_n = \lim_{n\to\infty}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right) = \frac{1}{e} \approx 0.36787944\cdots$

Remarks:

(i) The expectation of A is therefore the greatest in the case of one card and least in the case of two cards.

(ii) Again, when the number of cards is odd, the expectation of A is always greater than that for all even numbers of cards.

(iii) Thus, if the number of cards is even, then the expectation of A is less than that for all odd number of cards.

(iv) If the number of cards be infinite, the expectation of A will be expressed by $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \ldots$

In 1779, Euler presented [11] a paper on '*A curious question from the doctrine of combinations*'. This paper had some impact on the probability of winning a game i.e. on the lottery problems, where the number of combinations, ${}^nC_r$ of *n* objects selected *r* at a time with the order of the chosen objects is *not* taken into account and the number of permutations, ${}^nP_r$ of *n* objects taken *r* at a time with the order of chosen objects is taken into account.

4.2.1. Euler's Beta and Gamma Distributions:

Euler derived two important integrals viz. Beta and Gamma integrals also known as Beta function and Gamma function which play important roles in probability distribution theory.

4.2.1.1. Euler's Beta function:

Euler's Beta function is an improper integral denoted by $\beta(m,n)$, where *m* and *n* are two parameters of Beta function and defined as

$$\beta(m,n) = \int_0^\infty x^{m-1}(1-x)^{n-1}\,dx \qquad (8)$$

$$or \quad \beta(m,n) = \int_o^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \qquad (9)$$

Here equation (8) is known as Beta function of first kind and (9) is known as Beta function of second kind. It should be noted here that these two integrals are also known as special function. These two integrals have been used in probability distribution theory as follow:

4.2.1.2. Beta distribution of first kind:

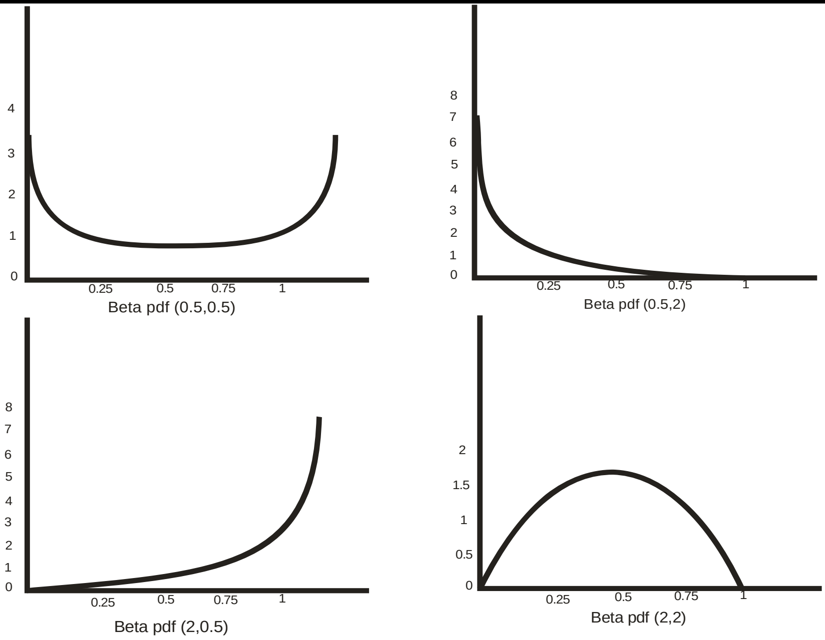
The continuous random variable '*x*' is said to follow Beta distribution of first kind with parameters *m* and *n* respectively, if its probability density function (pdf) is given by

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}, & (m,n) > 0, \quad 0 \le x \le 1 \\ 0, & otherwise \end{cases}$$

4.2.1.3. Beta distribution of second kind:

The continuous random variable '*x*' is said to follow Beta distribution of second kind with parameters *m* and *n* respectively, if its probability density function (pdf) is given by

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}}, & (m,n) > 0, \quad x > 0 \\ 0, & otherwise \end{cases}$$



4.2.1.4. Euler's Gamma function:

Euler's improper integral Gamma function is denoted by Γ and defined by

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

Here $n>0$ is the parameter.

Similarly it can be shown that $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}$

Here $a>0$ is another parameter of the Gamma function and it plays important role in probability distribution theory.

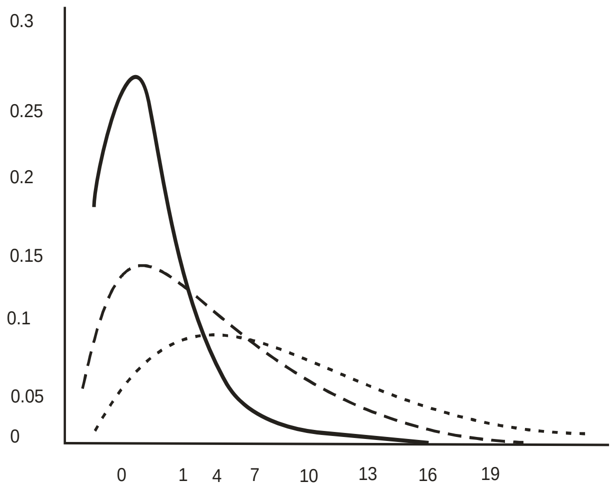
4.2.1.5. Gamma distribution:

The continuous random variable 'x' is said to follow Gamma distribution with parameters $\lambda > 0$ and $n > 0$, if its probability density function (pdf) is given by

$$f(x) = \begin{cases} \frac{e^{-\lambda x} \lambda^n x^{n-1}}{\Gamma(n)}, & x > 0, \quad (\lambda, n) > 0 \\ 0, & otherwise \end{cases} \tag{10}$$

It should be noted here that, if we take $n = 1$, in the above expression (10), then it reduces to

$f(x) = \lambda e^{-\lambda x}$, $x > 0$, $\lambda > 0$, which is the pdf of negative exponential or simply exponential distribution. Hence exponential distribution is a special case of Gamma distribution and it enjoys 'memory less' property. For different values of λ and n, we get the following type rough graph.



Gamma distribution pdf.

5. Contribution to Probability by Pierre -Simon Laplace:

5.1. Pierre- Simon Laplace's personal life:

Pierre Simon Laplace was born in 23rd March 1749 at Beaumont-en-Auge, a kingdom of France. He was a French Mathematicians whose work was important in the development of Engineering, Mathematics, Statistics and Physics. He summarized and extended the work of his predecessors in his five-volumes '*Celestial Mechanics*'. In Probability Theory the Bayesian Interpretation of Probability was developed by Laplace. He formulated 'Laplace's equation' and pioneered the 'Laplace transform'. The Laplacian differential operator is widely used in Mathematics. He re-stated and developed the Nebular Hypothesis of the origin of the Solar System. He was one of the first scientists to postulate the existence of Black Holes and the notion of gravitational collapse. Laplace is remembered as one of the greatest scientists of all time. He was Napoleon's examiner when Napoleon attended the École Militaire in Paris in 1784. Laplace became a 'Count of the Empire' in 1806 and was named as 'Marquis' in 1817, after the Bourbon Restoration. He died on 5th March 1827 in Paris, France.

Laplace's scientific works were concentrated mainly into two different areas, viz. Celestial Mechanics and Probability and Mathematical Statistics. In late 18th century and early 19th century, 'Normal distribution' was more explicitly recognized and appreciated for its role in describing errors of observations. Laplace is credited with the first proof of a Central Limit Theorem.

5.2. Contribution to Probability Theory by Pierre- Simon Laplace:

Laplace wrote around eighteen memoirs dealing with the Probability Theory [10]. In 1774, at the age of 25 years, Laplace wrote memoir '*Mémoiresur la probabilités des causes par les évènemens*'. In 1778 Laplace wrote another memoir '*Mémoiresur les probalités*' but it was published in 1781. In 1812 Laplace published another two memoirs '*Essai Philosophiquesur les Probabilités*' and '*Théorie Analytique des Probabilités*'. These two are among the best-known contributions of Laplace towards Probability Theory.

Laplace initially presented his idea of Inverse Probability in '*Mémoiresur la probabilités des causes par les évènemens*'. In the second memoir, '*Mémoiresur les probalités*', he dealt with more topics and was more elaborate in comparison to the first one [10]. In it, Laplace completed his application of probability calculus to the analyses of 4 errors in observations, which was left unfinished in the previous memoir. Laplace also introduced the 'principle of probabilistic inferences' in a more explicit manner.

5.2.1. Laplace's Inverse Probability [10]:

Laplace gave his presentation of Inverse Probability by first mentioning the difference between direct and indirect or inverse probability. He continued by stating the Problem of inverse probability for which he gave solutions in different formats: '*An urn is supposed to contain a given number of white and black tickets in an unknown ratio; if one draws a ticket and finds it white, determine the probability that the ratio of white and black tickets is that of p to q. The event is known and the cause is unknown.*' [Laplace, 1774]

In order to solve the problem, he defined the principle of inverse probability: '*If an event can be produced by a number n of different causes, the probabilities of these causes given the event are to each other as the probabilities of the event given the causes, and the probability of the existence of each of these is equal to the probability of the event given that cause, divided by the sum of all the probabilities of the event given each of these causes.*'

In modern mathematical form, Laplace's principle can be written as follows:

If the 'event' is denoted by E and $A_1, A_2, \ldots, A_n$ be the n potential causes, then

$$\frac{P(A_i/E)}{P(A_j/E)} = \frac{P(E/A_i)}{P(E/A_j)} \Rightarrow P(A_i/E) = \frac{P(E/A_i)}{\sum P(E/A_j)}$$

It is very clear that Laplace's principle has the same form as the Bayes' formula. In the second edition of '*Théorie Analytique des Probabilités*'(published in 1814), Laplace proved the general version of his principle, which is the same as Bayes' theorem.

Laplace gave the following example to illustrate the principle [10]: There are two urns, A and B. The first contains p white tickets and q black tickets, and the second contains p' white tickets and q' black tickets. One draws a white and b black tickets from either of these urns, but not knowing which of the urns. It is required to determine what is the probability that the urn where the tickets were drawn was A or B?

Laplace presented the following solution: Assuming that the urn was A, the probability of getting a white and b black tickets from it is

$$K = P\left(\frac{(a,b)}{A}\right) = \frac{\dfrac{(a+b)!(p+q-a-b)!}{a!b!(p-a)!(q-b)!}}{\dfrac{(p+q)!}{p!q!}}$$

The probability that the urn was B, i.e., $K' = P\left(\frac{(a,b)}{B}\right)$, can be obtained in a similar way replacing p and q by p' and q' respectively.

Applying the principle, Laplace concluded that the probability that the urn was A is $P(A) = \frac{K}{K+K'}$ and the probability that it was B is $P(B) = \frac{K'}{K+K'}$.

5.2.2. Laplace and Central Limit Theorem:

According to [1], Laplace is credited with the first proof of a Central Limit Theorem. Jerzy Neyman, in his text '*First Course in Probability and Statistics*', presented Laplace's Theorem (in contemporary notation) as follows:

Whatever be two numbers $t_1 < t_2$ and whatever be the fixed value of the probability of success p, $0 < p < 1$, if the number n of completely independent trials is indefinitely increased, then the probability that the corresponding binomial variable $X(n)$ will satisfy the inequalities $t_1 < \frac{X(n)-np}{\sqrt{np(1-p)}} < t_2$ tends to the limit $\frac{1}{\sqrt{2\pi}}\int_{t_1}^{t_2} e^{-\frac{x^2}{2}}dx$.

The normal distribution and central limit theorem have considerable application in the contemporary practice of probability and statistics.

5.2.3. Method of Least Squares:

In 1810, Laplace wrote "*if one were combining observations each one of which was itself the mean of a large number of independent observations, then the least squares estimates would not only maximize the likelihood function, considered as a posterior distribution, but also minimize the expected posterior error, all this without any assumption as to the error distribution or a circular appeal to the principle of the arithmetic mean*". In fact, Laplace wrote so after he saw Gauss's work. He showed that this result provided a Bayesian justification for least squares [6]:

If the mean value of a set of observations is the most probable value and the error function for the observations is of the Gaussian form:

$$\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$$

where h is known as the measure of precision or called the precision constant. The value of h should be chosen in a way which makes the probability maximum. If $x_1, x_2, \ldots, x_n$ are the measured observations of a number x, then the deviations of the observations from x are respectively $x-x_1,\ x-x_2,\ \ldots,\ x-x_n$. Assuming the probability $p(x-x_k)$, $k = 1, 2, 3, \ldots, n$ to be Gaussian, i.e. $p(x-x_k)=\frac{h}{\sqrt{\pi}}\exp\left(-h^2(x-x_k)^2\right)$, the probability of all deviations is the product

$$\prod_{k=1}^{n} p(x-x_k)=\frac{h^n}{\pi^{\frac{n}{2}}}\exp\left(-h^2\sum_{k=1}^{n}(x-x_k)^2\right) \qquad (11)$$

The problem is to find x so that probability is maximum which is equivalent to determining x so that $\sum_{k=1}^{n}(x-x_k)^2$ is a minimum. This gives the mean value $\bar{x}$ of $x_1, x_2, \cdots x_n$. This method of finding the best value of an observation by assuming that the sum of the squares of the deviations from it will be a minimum is called the Method of Least Squares. Putting the mean value $\bar{x}=\frac{1}{n}\sum_{1}^{n}x_n$ in the right hand side of (11), and making the above probability maximum, it turns out that h is determined by the equation [5]:

$$\frac{d}{dh}\left(h^n\exp\left(-h^2\sum_{k=1}^{n}(x-x_k)^2\right)\right)=0, \text{ so that } \frac{n}{h}=2h\sum_{k=1}^{n}\left(x-\bar{x}_k\right)^2.$$

Thus,
$$h^2=\frac{n}{2\sum_{k=1}^{n}(x-x_k)^2}=\frac{1}{2\sigma'^2}$$

Where σ' is the standard deviation for the given set of observations defined by $n\sigma'^2=\sum_{k=1}^{n}(x-\bar{x})^2$

It follows that the choice of h is such as to make the standard deviation for the observed set $x_1, x_2, \cdots x_n$ coincide with that of the given population.

6. Contribution to Probability by Johann Carl Friedrich Gauss:

6.1. Johann Carl Friedrich Gauss's personal life:

Johann Carl Friedrich Gauss was born on 30th April 1777 at Brunswick, Principality of Brunswick- Wolfenbüttel. Johann Carl Friedrich Gauss was a great German Mathematician. Gauss had an exceptional influence in many fields of Mathematics and Science. Gauss was a child prodigy. At the age of three years, he corrected an error in his father's payroll calculation. At the age of five years, he was looking after his father's accounts on regular basis. At the age of seven years he was reported to have amazed his teacher by summing the integers from 1 to 100 almost instantly. At the age of 14, the Duke of Brunswick granted Gauss a stipend and he entered the Collegium Carolinium, where he studied the works of Newton, Euler, and Lagrange and modern language. In 1795, he entered the University of Göttingen. Within a year, Gauss constructed a 17-sided regular polygon using only ruler and compass which had been the first of its kind constructed since antiquity. In 1798, at the age of 21 years, he completed his book '*Disquisitiones Arithmeticae*' which was published on 1801. This work laid the foundations for modern number theory. In 1798 Gauss returned to Brunswick and in 1799 he entered the University of Helmstedt, where he earned a doctorate degree. The dissertation '*Demonstratio nova.*', published in 1801, contained Gauss's first proof of the '*Fundamental Theorem of Algebra*'. In the same year, without publishing his method, Gauss correctly predicted the location of the first known asteroid, Ceres. Later on, his theoretical prediction was experimentally verified.

Gauss's contributions to statistics revolve around the conceptual convergence which is known as the 'Gauss – Laplace synthesis'. Since 1809, this foundational merger advanced effective methods for combining data with the ability to quantify error. Gauss's role in this synthesis centres in his contribution towards the theory of least squares, his use of the normal curve, and the influence this work had on Pierre Simon Laplace. During the mid-seventeenth century, astronomers wanted to know how best to combine a number of independent but varying observations of the same phenomenon. Among the most promising procedures was the method of least squares which argues that the minimal distance to the true value of a distribution of observations is the sum of squared deviations from the mean. In 1807, Gauss became director of the University of G¨ottingen's observatory. In 1845 he became an associated member of the Royal Institution of the Netherlands. As a mathematician and scientist, we can rank Gauss with Archimedes and Newton. On 23rd February 1855, he died of a heart attack in $G\ddot{o}ttingen$.

6.2. Contribution to Probability Theory by Johann Carl Friedrich Gauss:

Gauss's first approach to least squares is contained in his book '*Theoria Motus Corprum Coelestium*', and it emerged during his discussion of the calculation of planetary orbits on the basis of any number of observations. Despite early insights in to applications of least squares, Gauss failed to publish an account of it until 1809. Moreover, it was published only in the last chapter of a major contribution towards celestial mechanics. A priority dispute arose with Adrien Marie Legendre, who first published a least squares discussion in 1805. Even if publication priority related to the concept of least squares belongs to Legendre, Gauss offered a sophisticated elaboration of the method. It was Gauss, not Legendre, who developed the method into a statistical tool, embedding it into a statistical framework, involving the probabilistic treatment of observational errors, and thus set the famous linear model on its modern course. In addition to developing the concept of least squares, the 1809 publication contained another seminal contribution to statistics, that is, use of the normal distribution to describe measurement error. Here, Gauss employed Laplace's probability curve for sums and means to describe the measurement of random deviations around the true measurement of an astronomical event. Because of this insight, by the end of the nineteenth century, what we know today as the normal distribution came to be known as the Gaussian distribution. Although Gauss was not the first to describe the normal curve, he was the first to use it to assign precise probabilities to errors. In honour of this insight, the 10 Deutschmark was printed with both an image of Gauss and the normal curve's geometric and formulaic expression. Gaussian insights of 1809 proved a conceptual catalyst. In 1810, Laplace presented what we know today as the central limit theorem: '*the distribution of any sufficiently sampled variable can be expressed as the sum of small independent observations or variables approximating a normal curve*'. When Laplace read Gauss's 1809 book later that year, he recognized the connection between his theorem, the normal distribution, and least squares estimates. If errors are aggregates, then errors should distribute along the normal curve with least squares providing the smallest expected error estimate. The coming years were a conceptual watershed as additional work by Gauss, Laplace, and others converged to produce the Gauss–Laplace synthesis. As Youden ([12], p. 55) later observed "*The normal law of error stands out...as one of the broadest generalizations of natural philosophy... It is an indispensable tool for the analysis and the interpretation of the basic data obtained by observation and experiment*". Following 1809, these insights spread from astronomy to physics and also to other branches of science. Absorption of the idea into the social sciences took more time.

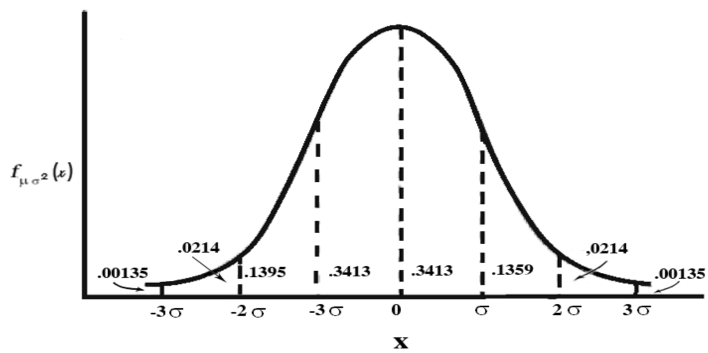
6.2.1 Gaussian distribution, Gaussian curve:

In the field of Probability Theory, Gauss introduced Gaussian Distribution, Gaussian Function and Gaussian Error Curve. He tried to show how probability could be represented by 'Normal' curve which peaks around the mean and quickly falls off towards infinity in positive and negative directions. The Gaussian Distribution is also described as a 'Bell shaped curve'.

If x be a continuous random variable and x follows a Gaussian distribution with arithmetic mean μ and standard deviation σ . Then, its probability density function is given by

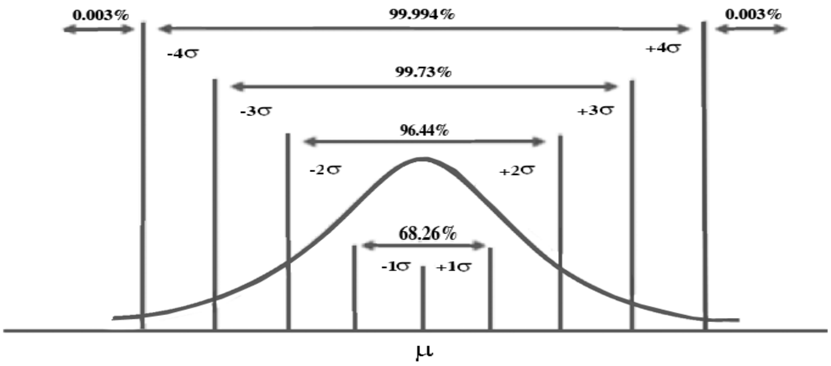
$$f_{\mu\sigma^2}(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}, \qquad x \in (-\infty, \infty)$$

The Normal or Gaussian curve of Probability Distribution is used in Statistics. In the following graph, μ is the mean and σ^2 is the variance.



According to [9], some researchers refer the Gaussian curve as '*Curve of Nature itself*' because of its inherent and versatile nature. Gaussian curve represents some phenomena that follows the Gaussian distribution. A few of those are mentioned below:

Duration of human pregnancy, Human blood glucose level, Amount of hemoglobin in men per 100 ml of blood, Body mass index of athletes, Height of men and women, Number of times an adult breathes per minute, Number of rain drops falling in a storm, University students' intelligence quotient, Life of automatic dishwashers etc.



In the above figure, it is noticed that 68.26% of the values of a Gaussian random variable are between a positive standard deviation and a negative standard deviation in relation to its arithmetic mean μ . Similarly, 95.44% of the values of a Gaussian random variable are between two positive and two negative standard deviations, 99.73% of the values of a Gaussian random variable are between three positive and three negative standard deviations and lastly 99.994% of the values of a Gaussian random variable are between four positive and four negative standard deviations in relation to the arithmetic mean μ . Values 0.006% that are spaced four deviations above and four deviations below the mean are called outliers.

6.2.2. Characteristics of Gaussian curve:

Gaussian curve has several characteristics as mentioned below [9]:

1. The Gaussian curve is a continuous random variable x with mean μ and standard deviation σ.
2. The curve is symmetric around the mean, that is, the graph is in the form of a bell.
3. The highest frequency value coincides with the value of the mean and the median.
4. The width of the curve is determined by the standard deviation. Larger values of standard deviation determine longer curves, showing the variability of the data.
5. The total area on the curve is one, that is, $\int_{-\infty}^{+\infty} f(x)\, dx = 1$, where $f(x)$ is the probability density function.
6. The probability $P(a \le x \le b) = \int_a^b f(x)\, dx$ is the area under the curve in the interval (a,b).
7. Most phenomena, in which there is a set of values with a random characteristic, have an approximately normal distribution (following a normal curve).

7. Contribution to Probability by Sime/on Denis Poisson:

7.1. Sime/on Denis Poisson's personal life:

Poisson was born in 21st June 1781, in Pithiviers, Loiret district in France. His father was an army officer. In 1818, he was elected a fellow of the Royal Society. In 1822, he became a foreign honorary member of the American Academy of Arts and Science. In 1823, he became a foreign member of the Royal Swedish Academy of Sciences. He is famous for his works: Poisson point process, Poisson's equation, Poisson kernel, Poisson distribution and many more. He died on 25th April 1840 at Sceaux, France. The main works of Poisson were devoted to Mechanics and Mathematics. In particular, he achieved much in the fields of Definite Integrals, Equations in Finite Differences, Partial Differential Equations, Mathematical physics and Probability.

7.2. Contribution to Probability Theory by Sime/on Denis Poisson:

7.2.1. Poisson concepts between the probability of an event and its chance:

According to [7] Poisson distinguished between the probability of an event and its chance. The former, as he understood, was subjective, and the latter was objective. Explaining his definition of subjective probability, Poisson noted in passing that with infinite m and n (e.g., if m and n are areas of certain figures) it could become irrational. Poisson here paid no attention to the inadequacy of the classical definition of probability. Thus, probability measured by the ratio of the number of favorable cases m to the total number of cases n could change with experience while chance is constant. Poisson first expounded his point of view in 1836, in a letter to Cournot, who later included it in his book entitled, '*Exposition de la théorie des chances et des probabilities*' (pp. vi-vii) which was published in 1843:

7.2.2.1. Poisson distribution:

A discrete random variable X is said to have a Poisson distribution with parameter $\lambda > 0$, if for $k = 0,1,2,\ldots$, the probability mass function of X is given by

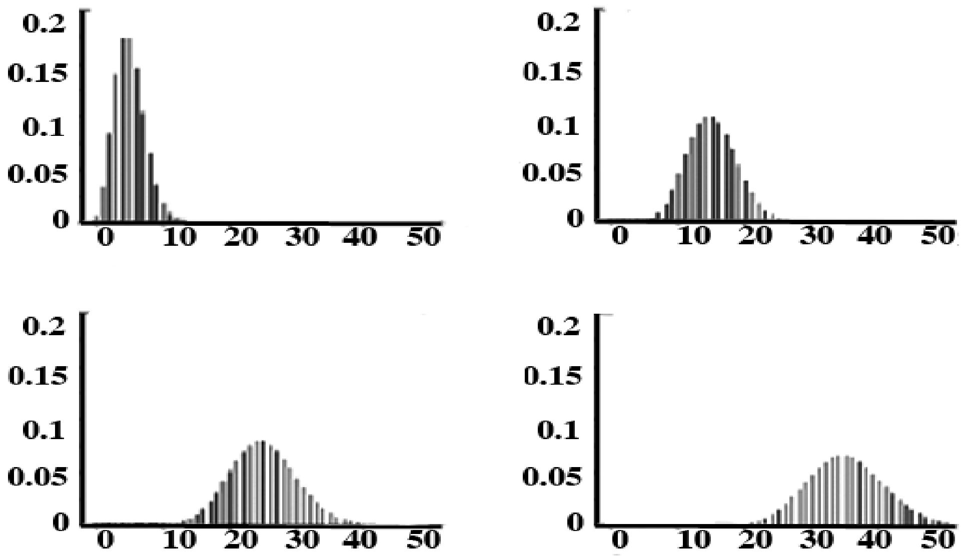
$$f(k,\lambda) = \Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Where e is the Euler's number, $e = 2.71828\ldots$, the positive real number λ is equal to the expected value of X and also to its variance i.e. $\lambda = E(X) = Var(X)$.

Distribution function of Poisson distribution can be expressed in terms of incomplete Gamma integral. Mathematically, we can express it as given below:

$$f(x) = \frac{1}{\Gamma(x+1)} \int_{\lambda}^{\infty} e^{-t} t^x dt , \qquad x = 0, 1, 2, \ldots$$

$$\text{where} \qquad \int_{\lambda}^{\infty} e^{-t} t^x dt = \frac{e^{-\lambda}\lambda^x}{x!} + \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} + \ldots + e^{-\lambda}\lambda + e^{-\lambda}$$



Poisson pmf (λ= 5, 15,25,35)

The Poisson distribution can be applied to model events such as the number of patients arriving in an emergency room between 1 to 2 pm, the number of photons hitting a detector in a particular time interval etc.

7.2.3. Poisson's Cumulative Distribution Functions:

Definition of a cumulative distribution function for a discrete random quantity was first formulated by Poisson as $F_n(x) = P\{x_n < x\}$, where n is the number of observations.

Moreover, Poisson [7] defined the density as the derivative of $F_n(x)$ and actually used this second definition even for continuous random quantities. Thus, Poisson's starting point was the cumulative function rather than the density function. Poisson also introduced the cumulative distribution function for continuous random quantities.

8. Conclusion:

In this paper, we have tried to collect some of the prominent contributions of a few selected great European Mathematicians towards Probability Theory which has now become a very important topic in itself having applications in many scientific disciplines. We hope, further research in this line will definitely help the future researchers to have a glimpse about what a great contribution these great European Mathematicians made towards the development of Probability Theory in particular and scientific field in general.

REFERENCES

[1]. Daniel McFadyen: An Historical Survey of the Development of Probability and Statistics based on the Appearance of Fundamental Concepts, September 8, 2003**.**

[2]. O. B. Sheynin: S. D. Poisson's Work in Probability, Archive for History of Exact Sciences, Vol. 18, No. 3 (15.III.1978), pp. 245-300, Published by Springer.

[3]. Ivo Schneider: Direct and indirect influences of Jakob Bernoulli's Ars Conjectandi in 18th century Great Britain, Electronic Journ@l for History of Probability and Statstics, vol2, June/2006.

[4]. Bruno A. Olshausen : Bayesian probability theory, March 1, 2004.

[5]. Lokenath Debnath (Department of Mathematics, University of Texas-Pan American, Edinburgand) Kanadpriya Basu (Department of Mathematical Sciences University of Texas at El Paso) : A Short History of Probability Theory and Its Applications, Article in International Journal of Mathematical Education, January 2015.

[6]. Stephen M. Stigler: Studies in the History of Probability and Statistics. XXXIV Napoleonic statistics: The work of Laplace ; Department of Statistics, University of Wisconsin, Madison, Biometrika, Vol. 62, No. 2 (Aug., 1975), pp. 503-517.

[7]. O. B. Sheynin : S. D. Poisson's Work in Probability, Archive for History of Exact Sciences, Vol. 18, No. 3 (15.III.1978), pp. 245-300, Published by: Springer.

[8]. Leonard Euler: M´emoires de l'acad´emie de Berlin (1751), (Calculus of the probability in the game of Rencontre, 1753), (pp. 255-270.)

[9]. Edel Alexandre Silva Pontes (Department of Mathematics, Federal Institute of Alagoas, Brazil): A Brief Historical Overview Of the Gaussian Curve: From Abraham De Moivre to Johann Carl Friedrich Gauss., International Journal of Engineering Science Invention (IJESI) ISSN (Online): 2319 – 6734, ISSN (Print): 2319 – 6726, Volume 7, Issue 6 Ver V, June 2018(pp. 28-34).

[10]. Vesa Kuusela: Laplace- a pioneer of statistical inference**,** Electronic Journ@l for History of Probability and Statistics, Vol8; December 2012.

[11]. Lokenath Debnath: The Legacy of Leonhard Euler, A Tricentennial Tribute, pages 337-348, Published by Imperial College Press 57 , Shelton Street ,Covent Garden, London WC2H 9HE.

[12]. Youden, W.J. (1994): Experimentation and Measurement, U.S. Department of Commerce, Washington.