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OPTIMIZATION OF MULTIPLE SERVER INTERDEPENDENT QUEUEING MODELS WITH BULK SERVICE

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Abstract: In this paper it is devoted to multiple server queuing models having bulk service with interdependence .First the two server interdependent queuing model with bulk service is developed and then extended to multiple server case. In both the models the explicit expression are obtained for the steady state system characteristics to analyze their behavior.

Keywords: **The system characteristics like, mean queue length, variability of the system size, coefficient of variation are derived and analyzed in the light of the dependence parameter. These models also include the earlier models as particular cases for specific values of the parameters.**

I. INTRODUCTION

In this paper, two server interdependent queuing model with bulk service is considered for analysis. we assumed that the bulk service is process is having two service facilities with capacities b_1 and b_2. A batch of b_1 units or the whole queue length whichever is smaller is taken from the head of the queue for service in the first channel whenever it is free. Similarly, the second channel on becoming free takes b_2 (δ b_1) or the whole queue length whichever is less. If both the servers are idle and there is no queue the next unit to arrive always goes to the first service facility. Here, we assume the two service facilities are independent of each other and the arrival and service processes are interdependent. These sorts of situations are more common in marshalling yards with two engines, elevator process with two lifts etc. This model is also extended to multiple server interdependent queuing models with bulk service.
In both the models we first develop the difference-differential equations and solve them through generating function techniques. The system characteristics are derived and analyzed in the presence of the dependence parameter .These models can be viewed as generalizations of the Bulk Service queuing models given by Arora (1963)

II. TWO SERVER INTERDEPENDENT QUEUEING MODEL WITH BULK SERVICE

In this paper , the two server queuing model with bulk service having interdependent arrival and service processes is considered, The two service facilities function independent of each other . We assume the arrivals are poissonian with the mean arrival rate λ .The marginal distributions of the service time in the two service facilities are exponential with the mean service rates μ_1 and μ_2. respectively. Then the conditional process of the number of service completions of the first service facility given that the number of arrivals is of the form,

$$P(X_{11} = n_1 / X_2 = n_2 ;\ t) = -(\mu_1 - \epsilon_1)t \sum_{j=0}^{min(n_1,n_2)} \binom{n_1}{j} \left(\frac{\epsilon_1}{\lambda}\right) \left(\frac{\lambda - \epsilon_1}{\lambda}\right)^{n_2 - j} \frac{[(\mu_1 - \epsilon_1)t]^{n_1 - j}}{(n_1 - j)!} \quad \ldots\ldots (1.2.1)$$

Where
X_{11} is the number of service completions of the first service facility during time t.
X_2 is the number of arrivals during time t.
$\in$ is the mean dependence rate.

Similarly, the conditional process of the number of service completions of the second service facility given the number of arrivals is of the form

$$P(X_{12} = n_1 / X_2 = n_2;\ t) = e^{-(\mu_2-\epsilon_2)t} \sum_{j=0}^{min(n_1,n_2)} \binom{n_2}{j} \left(\frac{\epsilon_2}{\lambda}\right)^j \left(\frac{\lambda-\epsilon_2}{\lambda}\right)^{n_2-j} \frac{[(\mu_2-\epsilon_2)t]^{n_1-j}}{(n_1-j)!} \quad \ldots(1.2.2)$$

Where

X_{12} is the number of service completions of the second service facility during time t.

X_2 is the number of arrivals during time t.

$\in$ is the mean dependence rate.

Let $P_{00}(t)$ be the probability that at time t, both the channels are empty and there is no unit waiting in the queue. $P_{10}(t)$ Probability that at time t, the first channel is busy and the second channel is empty and no unit is waiting in the queue. $P_{01}(t)$ be the probability that at time t, the first channel is empty and the second channel is busy and no unit is waiting in the queue $P_{11}(t) = P_{10}(t) + P_{01}(t)$ be the probability that at time t, either of two channels is busy and no unit is waiting in the queue and $P_n(t)$ be the probability that at time t, both the channels are busy and there $n(\geq 0)$ units waiting in the queue.

With the above probabilities, the difference-differential equations of the model are

$$P'_{00}(t) = -(\lambda - \in)P_{00}(t) + (\mu_1 - \epsilon_1)P_{10}(t) + (\mu_2 - \epsilon_2)P_{11}(t)$$

$$P'_{10}(t) = -(\lambda + \mu_1 - \in - \epsilon_1)P_{10}(t) + (\lambda - \in)P_{00}(t) + (\mu_2 - \epsilon_2)P_0(t)$$

$$P'_{01}(t) = -(\lambda + \mu_2 - \in - \epsilon_2)P_{01}(t) + (\mu_1 - \epsilon_1)P_0(t)$$

$$P'_0(t) = -(\lambda + \mu_1 + \mu_2 - 2\in)P_0(t) + (\mu_1 - \epsilon_1)\sum_{i=1}^{b_1} P_i(t) + (\mu_2 - \epsilon_2)\sum_{j=1}^{b_2} P_j(t) + (\lambda - \in)[P_{10}(t) + P_{01}(t)]$$

$$P'_n(t) = -(\lambda + \mu_1 + \mu_2 - 2\in)P_n(t) + (\lambda - \in)P_{n-1}(t) + (\mu_1 - \epsilon_1)P_{n+b_1}(t) + (\mu_2 - \epsilon_2)P_{n+b_2}(t)$$
$$for\ n \geq 1$$

Where $\in = \epsilon_1 + \epsilon_2$ …………..(1.2.3)

With the initials conditions $P_{00}(0) = 1$, $P_{10}(0) = 0$, $P_{01}(0) = 1$ and $P_n(0) = 0$

Let $F(x,t)$ be the generating function of $P_n(t)$

$$F(x,t) = \sum_{n=0}^{\infty} P_n(t)\, x^n \quad \ldots\ldots\ldots.(1.2.4)$$

Multiplying equation (1.2.3) by proper powers of x and add, we get

$$\frac{\delta F(x,t)}{\delta t} = -\left\{(\lambda + \mu_1 + \mu_2 - 2\in) - (\lambda - \in)x - \frac{(\mu_1 - \epsilon_1)}{x^{b_1}} - \frac{(\mu_2 - \epsilon_2)}{x^{b_2}}\right\}F(x,t) + (\mu_1 - \epsilon_1)\sum_{i=0}^{b_1-1}(1 - x^{i-b_1})\,P_i(t) + (\mu_2 - \epsilon_2)\sum_{j=0}^{b_2-1}(1 - x^{i-b_2})\,P_j(t) - (\mu_1 + \mu_2 - \in)P_0(t) + (\lambda - \in)[P_{10}(t) + P_{01}(t)] \quad \ldots\ldots\ldots.(1.2.5)$$

Applying Laplace-Transformation to the equations (1.2.3) and (1.2.5) and using the initial conditions, we have

$$-(s + \lambda - \in)P^*_{00}(t) + (\mu_1 - \epsilon_1)P^*_{10}(t) + (\mu_2 - \epsilon_2)P^*_{01}(t) = 0$$

$$-(s + \lambda + \mu_1 - \in - \epsilon_1)P^*_{10}(t) + (\lambda - \in)P^*_{00}(t) + (\mu_2 - \epsilon_2)P^*_0(t) = 0$$

$$-(s + \lambda + \mu_2 - \in - \epsilon_2)P^*_{01}(t) + (\mu_2 - \epsilon_2)P^*_0(t) = 0 \quad \ldots\ldots\ldots.(1.2.6)$$

$$F^*(x,s) = x^{b_1}\{(\mu_1 - \epsilon_1)\sum_{i=0}^{b_1-1}(1 - x^{i-b_1})\,P^*_i(s) + (\mu_2 - \epsilon_2)\sum_{j=0}^{b_2-1}(1 - x^{i-b_2})\,P^*_j(s) - (\mu_1 + \mu_2 - \in)P^*_0(s) + (\lambda - \in)[P^*_{10}(s) + P^*_{01}(s)]\} / \{-(\lambda - \in)x^{b_1+1} + (s + \lambda + \mu_1 + \mu_2 - 2\in)x^{b_1} - (\mu_1 + \mu_2)x^{b_1+b_2} + (\mu_1 - \epsilon_1)\} \quad \ldots\ldots\ldots.(1.2.7)$$

Applying Roche's theorem, it can be seen that for b_2

$$(\lambda - \in)x^{b_1+1} - (s + \lambda + \mu_1 + \mu_2 - 2\in)x^{b_1} - (\mu_2 - \epsilon_2)x^{b_1-b_2} + (\mu_1 - \epsilon_1) = 0 \quad \ldots\ldots\ldots(1.2.8)$$

has b_1 zero's inside and outside $|x| = 1$ .Let the zero's that lies inside $|x| = 1$ be denoted by

$x_k(s)$, $k = 1,2, \ldots \ldots b$. and that which lies outside be $x_0(s)$.

Since $F^*(x,s)$ is regular inside $|x| = 1$ and the denominator has got b_1 , zeros inside the unit circle . The numerator must vanish at those zeros giving rise to b linear equations.

They are

$(\mu_1-\epsilon_1)\sum_{i=0}^{b_1-1}\left(1-x_k{}^{i-b_1}(s)\right)P^*{}_i(s)+(\mu_2-\epsilon_2)\sum_{j=0}^{b_2-1}\left(1-x_k{}^{j-b_2}(s)\right)P^*{}_j(s)-(\mu_1+\mu_2-\in)P^*{}_0(s)+(\lambda-\in)[P^*{}_{10}(s)+P^*{}_{01}(s)]=0$ For k = 1,2,……………..,b_1 ………….(1.2.9)

In (b_1+2) unknowns. Out of these, $P^*{}_{10}(s)$ and $P^*{}_{01}(s)$ could be expressed in terms of $P^*{}_0(s)$. From equation (1.2.6) and have these set of equations (1.2.6) would contains unknowns in $P^*{}_n(s), n=0,1,2,\dots\dots,(b-1)$ and as such the unknowns would be determined completely. We can express $F^*(x,s)$, since the degree of the denominator is more than the numerator by one, in terms of the exterior zero as.

$$F^*(x,s)=\frac{A(s)}{x_0(s)-x} \qquad \text{……….(1.2.10)}$$

Therefore,

$$P^*{}_n(s)=\frac{A(s)}{x_0{}^{n+1}(s)} \qquad for\; n\geq 0$$

$$P^*{}_0(s)=\frac{A(s)}{x_0(s)} \qquad \text{…… ….(1.2.11)}$$

Here A(s) is an unknown to be determined. Solving equation (1.2.6) using equation (1.2.11)and writing $x_0(s)$ as x_0, we get

$P^*{}_{00}(s)=\{(s+\lambda-\in-\epsilon_1)+(\mu_1-\epsilon_1)(\mu_2-\epsilon_2)A(s)[2(s+\lambda-\in)+(\mu_1+\mu_2-\in)][x_0(s+\lambda-\mu_2-\in-\epsilon_2)]^{-1}\}/\{(s+\lambda-\in)(s+\lambda+\mu_1-\in-\epsilon_1)-(\lambda-\in)(\mu_1-\epsilon_1)\}$ ……..(1.2.12)

$P^*{}_{10}(s)=\{(\lambda-\in)+(\mu_2-\epsilon_2)A(s)[(s+\lambda-\in)(s+\lambda-\mu_2-\in-\epsilon_2)+(\lambda-\epsilon_1)(\mu_1-\epsilon_1)][x_0(s+\lambda-\mu_2-\in-\epsilon_2)]^{-1}\}/\{(s+\lambda-\in)(s+\lambda+\mu_1-\in-\epsilon_1)-(\lambda-\in)(\mu_1-\epsilon_1)\}$

………(1.2.13)

$$P^*{}_{10}(s)=(\mu_1-\epsilon_1)A(s)[x_0(s+\lambda-\mu_2-\in-\epsilon_2)]^{-1} \qquad \text{……...(1.2.14)}$$

Putting x=1 in equations (1.2.7) and (1.2.10) we get ,

$$\frac{A(s)}{x_0-1}=F^*(x,s)=\{-(\mu_1+\mu_2-\in)P^*{}_0(s)+(\lambda-\in)[P^*{}_{10}(s)+P^*{}_{01}(s)]\}/s \qquad \text{……..(1.2.15)}$$

Substituting equations (1.2.11) , (1.2.13) and (1.2.14) in (1.2.15) , we get

$A(s)=[(\lambda-\in)^2x_0(x_0-1)(s+\lambda-\mu_2-\in-\epsilon_2)]\,/$

$\{sx_0(s+\lambda-\mu_2-\in-\epsilon_2)[(s+\lambda-\in)(s+\lambda+\mu_1-\in-\epsilon_1)-(\lambda-\in)(\mu_1-\epsilon_1)]-(x_0-1)[(\lambda-\in)(s+\lambda-\in)\{(\mu_1-\epsilon_1)(s+\lambda+\mu_1-\in-\epsilon_1)(\mu_2-\epsilon_2)(s+\lambda-\mu_2-\in-\epsilon_2)\}]-(\lambda-\in)^2(\mu_1-\epsilon_1)(\mu_1-\epsilon_1+\mu_2-\epsilon_2)-(\mu_1-\epsilon_1+\mu_2-\epsilon_2)(s+\lambda-\mu_2-\in-\epsilon_2)[(s+\lambda-\in)(s+\lambda+\mu_1-\in-\epsilon_1)-(\lambda-\in)(\mu_1-\epsilon_1)]\}$ ………..(1.2.16)

Substituting the value of $A(s)$ from equation (1.2.16) into equations (1.2.11) to (1.2.14) gives the Laplace- Transform of all the probabilities defined earlier. To simplify the mathematical complexity, we take $\mu_1=\mu_2=\mu$ and $\epsilon_1=\epsilon_2=\in/2$, then equation (1.2.11) reduces to

$P^*{}_n(s)=(\lambda-\in)^2(x_0-1)/\left\{sx_n^n\left[(s+\lambda-\in)\left(s+\lambda+\mu-3\frac{\in}{2}\right)x_0-(\lambda-\in)\left(\mu_1-\frac{\in}{2}\right)\right]+(2\mu-\in)(x_0-1)\left(s+\lambda+\mu-3\frac{\in}{2}\right)\right\}$ ……...(1.2.17)

In the steady state if L_q is the mean number of customers in the queue then by using the tauberian theorem, namely

$$s\overset{lim}{\longrightarrow}0\; s\;\; P^*{}_n(s)=t\overset{lim}{\longrightarrow}\infty\, P_n(t)$$

Assuming the limit on the right exit , we have

$$L_q=\sum_{n=1}^{\infty}n\{sP^*{}_n(s)\}_{s=0} \;=\; 2\left(\frac{\lambda-\in}{2\mu-\in}\right)^2x_0\Big/\left\{\left[2\left(\frac{\lambda-\in}{2\mu-\in}\right)^2x_0+\left(1+\frac{2(\lambda-\in)}{2\mu-\in}\right)(x_0'-1)\right](x_0'-1)\right\} \qquad \text{……….(1.2.18)}$$

Where x_0 is the root of equation given in equation (1.2.8) , when $b_1=b_2=1$ then

$x_0'=2\frac{\mu}{\lambda}$ And the mean queue length is $L_q=\frac{2(\lambda-\in)^3(2\mu-\in)}{(2\mu-\in)^2-4(\lambda-\in)^2}$ ………(1.2.19)

Values of L_q are computed for various values of $\in$ and for fixed values of λ and b_1 and are given in table (1.1). From table (1.1) and equation (1.2.19) , it is observed that the values of L_q are decreasing as $\in$ increases and for fixed values of λ , μ and b_1, b_2 . The values of L_q decreases as b_2 increases and for fixed values of λ , μ and b_1, and the dependence parameter.

TABLE 1.1

VALUES OF $\mathbf{L_q}$

$b_1 = 5$, $\lambda = 2$ and $\mu = 3$

$b_1/\in$	0	0.2	0.4	0.6	0.8
0	0.1229	0.10132	0.08901	0.06181	0.0446
1	0.06056	0.0497	0.03941	0.03002	0.0216
2	0.05239	0.04337	0.03476	0.02669	0.01941
3	0.05081	0.044219	0.03391	0.026139	0.0191
4	0.05048	0.04192	0.03341	0.02603	0.0190

III. Busy Period Analysis

We determine the busy period distribution for two cases, namely

I. At least one of the server remains busy and
II. Both the servers remain busy

One channel is busy: The busy period starts with the arrival of unit when both the channels are empty and this lasts up to the instant at which both channels are become idle. We assume

$b_1 = b_2 = b$

$\mu_1 = \mu_2 = \mu$

$\epsilon_1 = \epsilon_2 = \in/2$

And $P_{11}(t) = P_{10}(t) + P_{01}(t)$

The required probability density is given by

$\lambda_1(t) = \frac{d}{dt} P_{00}(t)$ ………….(1.3.1)

and the corresponding equations are

$\frac{d}{dt} P_{00}(t) = (\mu - \in/2) P_{11}(t)$ ………….(1.3.2)

$F^*(x,s) = x^s\{((2\mu{-}\in)) \sum_{i=0}^{b-1}(1 - x^{i-s}) P_i^*(s) - (2\mu{-}\in)P_0^*(s) + (\lambda{-}\in)P_{11}^*(s)\}/$
$\{(\lambda{-}\in)x^{s+1} - (s + \lambda + 2\mu - 2\in)x^s + (2\mu{-}\in)\}$ ………….(1.3.3)

As before, let

$F^*(x,s) = B(s)/(x_0 - x)$

And $\frac{B(s)}{(x_0-1)} = F^*(1,s) = [(\lambda{-}\in)P_{11}^*(s) - (2\mu{-}\in)P_0^*(s)]/s$

Substituting the value of

$P_{11}^*(s) = \left(s + \lambda + \mu - 3\frac{\in}{2}\right)^{-1} [1 + (2\mu{-}\in)P_0^*(s)]$

$P_0^*(s) = \frac{B(s)}{x_0}$

In the above relation and solving for (s) , we get

$B(s) = (\lambda{-}\in)x_0(x_0 - 1)/\left[sx_0\left(s + \lambda + \mu - 3\frac{\in}{2}\right) + (2\mu{-}\in)(x_0 - 1)\left(s + \mu - \frac{\in}{2}\right)\right]$ ………….(1.3.4)

Then ,

$\lambda_1^*(s) = sP_{00}^*(s) = \left(\mu - \frac{\in}{2}\right) P_{11}^*(s)$

$= [(2\mu{-}\in)(s + 2\mu{-}\in)x_0 - (2\mu{-}\in)^2]/2\left[\left\{s\left(s + \lambda + \mu - 3\frac{\in}{2}\right) + (2\mu{-}\in)\left(s + \mu - \frac{\in}{2}\right)\right\}x_0 - (2\mu{-}\in)\left(s + \mu - \frac{\in}{2}\right)\right]$ ………(1.3.5)

This can be expressed as

$$\lambda_1^*(s) = \frac{(2\mu-\epsilon)(s+2\mu-\epsilon)x_0-(2\mu-\epsilon)^2}{2\left\{\left(s+\lambda+\mu-3\frac{\epsilon}{2}\right)+(2\mu-\epsilon)\left(s+\mu-\frac{\epsilon}{2}\right)\right\}x_0}\sum_{i=0}^{\infty}\left[\frac{(2\mu-\epsilon)\left(s+\mu-\frac{\epsilon}{2}\right)}{\left\{\left(s+\lambda+\mu-3\frac{\epsilon}{2}\right)+(2\mu-\epsilon)\left(s+\mu-\frac{\epsilon}{2}\right)\right\}x_0}\right]^i \quad \text{……….(1.3.6)}$$

Where x_0 is the root of the equation?

$$(\lambda-\epsilon)x^{s+1} - (s+\lambda+2\mu-2\epsilon)x^s + (2\mu-\epsilon) = 0 \quad \text{such that } |x_0| > 1 \quad \text{……….(1.3.7)}$$

Following the heuristic argument of Arora (1963) the moments of distribution are obtained by differentiating equation (1.3.5) and setting s=0.
The mean and variance are obtained as

Mean= $\left[\left\{1+\frac{\lambda-\epsilon}{2\mu-\epsilon}\right\}y_0 - 1\right] / \left[\left(\mu-\frac{\epsilon}{2}\right)(y_0-1)\right]$

$$\text{Variance}= \left\{\frac{\lambda-\epsilon}{2\mu-\epsilon}(s+1)\left[1+\frac{3(\lambda-\epsilon)}{2\mu-\epsilon}+\left(\frac{\lambda-\epsilon}{2\mu-\epsilon}\right)^2\right]y_0^3 - \left(1+\frac{\lambda-\epsilon}{2\mu-\epsilon}\right)\left[s\left(1+\frac{5(\lambda-\epsilon)}{2\mu-\epsilon}+\left(\frac{\lambda-\epsilon}{2\mu-\epsilon}\right)^2\right)+\frac{2(\lambda-\epsilon)}{2\mu-\epsilon}\right]y_0^2 + \left[\frac{\lambda-\epsilon}{2\mu-\epsilon}(s+2)+2s\left(1+\frac{\lambda-\epsilon}{2\mu-\epsilon}\right)^2\right]y_0 - \left(1+\frac{\lambda-\epsilon}{2\mu-\epsilon}\right)s\right\} \left\{\left(\mu-\frac{\epsilon}{2}\right)^2(y_0-1)^2\left[\frac{\lambda-\epsilon}{2\mu-\epsilon}(s+1)y_0-\left(1+\frac{\lambda-\epsilon}{2\mu-\epsilon}\right)s\right]\right\}^{-1}$$

Where $y_0 > 1$ is the root of equation………. (1.3.7)

In particular case when $s = 1$ the mean and variance of the busy period distribution can be obtained by setting $s = 1$ in equation. Thus ,

Mean $= \frac{2}{2\mu-\lambda}$ and Variance $= \frac{4(2\mu-\epsilon)}{(2\mu-\lambda)^3}$ ……….(1.3.8)

Both channels are busy: The length of the busy period is the length of the interval between the arrival of a unit that makes both the channels busy and the first subsequent moment at which one channel becomes empty.

The probability density for this case is

$\gamma_2(t) = \frac{d}{dt}V(t)$

The equation which determines the probability density are

$d\, P_{11}(t) = \left(\mu-\frac{\epsilon}{2}\right)P_0(t)$

$d\ \ P_0(t) = (\lambda+2\mu-2\epsilon)P_0(t) + (2\mu-\epsilon)\sum_{i=0}^{\infty}P_i(t)$

$\frac{d}{dt}P_n(t) = -(\lambda+2\mu-2\epsilon)P_n(t) + (\lambda-\epsilon)P_{n-1}(t) + (2\mu-\epsilon)P_{n+s}(t)$ For $n \in I$ ……….(1.3.9)

Under the initial conditions, the Laplace –Transform of $F(x,t)$ is

$F^*(x,s) = \{x^s + (2\mu-\epsilon)\sum_{i=0}^{s-1}(x^s-x^i)P_i^*(s) - (2\mu-\epsilon)x^sP_0^*(s)\} / -\{(\lambda-\epsilon)x^{s+1} - (s+\lambda+2\mu-2\epsilon)x^s + (2\mu-\epsilon)\}$….(1.3.10)

Following the same argument as given in previous section,

$F^*(x,s) = \frac{c(s)}{(x_0-x)}$ ……. (1.3.11)

Letting $x = 1$ in equation (1.3.10) and solving for $c(s)$, we get

$c(s) = x_0(x_0-1)/(s+2\mu-\epsilon)x_0(2\mu-\epsilon)$ ……. (1.3.12)

Thus,

$\overline{\gamma_2}(s) = (2\mu-\epsilon)(x_0-1)/[(s+2\mu-\epsilon)x_0-(2\mu-\epsilon)]$ …. (1.3.13)

Mean $= \frac{2}{2\mu-\lambda}$ and Variance $= \frac{(s+2\mu-\epsilon)(2\mu-\epsilon)}{(2\mu-\lambda)^3}$ ……….(1.3.14)

When $b_1 = 1, b_2 = 0, \mu_1 = \mu_2$ and $\epsilon = 0$ this model reduces to that of classical $M/M/1$ model.
This model is equivalent to the model (1.2) when b_2=0

IV. MULTI SERVER INTERDEPENDENT QUEUEING MODEL WITH BULK SERVICE

In this paper, we consider multiserver interdependent queueing model which is generalization of the model discussed in section (1.2) . As in the previous case here also we assume that the number of service completions of batches in each channels and number of arrivals are correlated and follows a bivariate Poisson distribution with parameters λ, μ and ϵ as mean arrival rate , mean service rate of each channel and mean dependence rate of each channel respectively.
The c μ is the composite service rate c ϵ is the composite dependence rate.
Let $P_{m,n}(t)$ be the probability that at time t , there are n units waiting in the queue and m channels are busy.
Following the heuristic argument of Ghare (1967) , the difference-differential equations of the model are

$P_{c,n}'(t) = -\left(\lambda - c\,\epsilon + c(\mu-\epsilon)\right)P_{c,n}(t) + (\lambda - c\,\epsilon)P_{c,n-1}(t) + c(\mu-\epsilon)P_{c,n+b}(t)$

$P'_{c,0}(t) = -(\lambda - c \in + c(\mu - \in))P_{c,0}(t) + c(\mu - \in)\sum_{k=1}^{b} P_{c,k}(t) + (\lambda - (c-1) \in)P_{c-1,0}(t)$

$P_{m,0}(t) = -(\lambda - m \in + m(\mu - \in))P_{m,0}(t) + (\lambda - (m-1) \in)P_{m-1,0}(t) + (m+1)(\mu - \in)P_{m+1,0}(t)$ For $1 \le m < c$

$P'_{0,0}(t) = -(\lambda - \in)P_{0,0}(t) + (\mu - \in)P_{1,0}(t)$ …………..(1.4.1)

Let $F(x,t)$ be the generating function of $P_{c,n}(t)$

$F(x,t) = \sum_{k=0}^{n} P_{c,k}(t)x^k$ …………..(1.4.2)

With the initial condition $P_{c,k}(0) = 1$, we have

$F(x,0) = 1$

Multiplying the first two equations of (1.3.1) appropriate powers of x and summing over all n , we get

$F'(x,t) = -\left[(\lambda - c \in) - (\lambda - c \in)x + c(\mu - \in) - \frac{c(\mu - \in)}{x^b}\right] F(x,t) \ + c(\mu - \in)\sum_{k=0}^{b-1}(1 - x^{k-b})P_{c,k}(t) - c(\mu - \in)P_{c,0}(t) + (\lambda - c \in)P_{c-1,0}$ …………..(1.4.3)

Taking the Laplace – Transform of equation (36) and further simplification we get ,

$F^*(x,s) = \left[1 + c(\mu - \in)\sum_{k=0}^{b-1}(1 - x^{k-b}){P^*}_{c,k}(s) - c(\mu - \in)P_{1,0}(s) + (\lambda - c \in){P^*}_{c-1,0}(s)\right] / \left[s + (\lambda - c \in) - (\lambda - c \in)x + c(\mu - \in) - \frac{c(\mu - \in)}{x^b}\right]$ …………..(1.4.4)

The zeros of the denominator of the equation (1.4.4) can be obtained from the solution to the equation

$\left[s + (\lambda - c \in) - (\lambda - c \in)x + c(\mu - \in) - \frac{c(\mu - \in)}{x^b}\right] = 0$ …………..(1.4.5)

Applying Roche's theorem to equation (1.4.5) be solution inside and one solution outside

$|x| = 1$. Let the ratio $F^*(x,s)$ can be expanded in terms of the external solution x_0 as.

$F^*(x,s) = \frac{A}{(x_0 - x)}$

And Therefore

${P^*}_{c,n}(s) = \frac{A}{{x_0}^{n-1}}$

${P^*}_{c,0}(s) = \frac{A}{x_0}$ …………..(1.4.6)

Where A is an unknown to be determined

Let ${P^*}_{m,0}(s)$ be the Laplace Transform of $P_{m,0}(t)$, then

$(\mu - \in){P^*}_{1,0}(t) = (\lambda - \in + s){P^*}_{0,0}(t)$

And $(m+1)(\mu - \in){P^*}_{m+1,0}(t)[(\lambda - m \in) + s + m(\mu - \in)]{P^*}_{m,0}(t)$ …………..(1.4.7)

The general solution of this set of equations can be written as

${P^*}_{m,0}(t) = {P^*}_{0,0}(t)\sum_{j=0}^{m}\prod_{i=1}^{m-j}\left[\frac{\lambda - i\in}{\mu - \in}\right]\left\{\Gamma\left(\frac{s}{\mu - \in}\right)\right\}/\left\{\Gamma\left(\frac{s}{\mu - \in}\right) j!\,(m-j)!\right\}$ …………..(1.4.8)

Define $\emptyset_m(s)$ as

$\emptyset_m(s) = \left\{\prod_{i=1}^{m}\left[\frac{\lambda - i\in}{\mu - \in}\right]\right\}\frac{1}{m!}2F_0\left(-m, \frac{s}{\mu}, -\frac{\mu - \in}{\lambda - \in}\right)$ …………..(1.4.9)

Where $2F_0$ is a generalized hypergeometric function

${P^*}_{c-1,0}(t) = {P^*}_{c,0}(t)\sum_{j=0}^{c}\prod_{i=1}^{c-1-j}\left[\frac{\lambda - i\in}{\mu - \in}\right]\left\{\Gamma\left(\frac{s}{\mu - \in + j}\right)\right\}/\left\{\Gamma\left(\frac{s}{\mu - \in}\right) j!\,(c-1-j)!\right\}$ …………..(1.4.10)

Rewriting equation (1.4.4) at x=1 and substituting equations (1.4.6) and (1.4.10) we get

$F^*(1,s) = \left\{1 - c(\mu - \in)\left(\frac{A}{x_0}\right) + (\lambda - \in)\ {P^*}_{0,0}(t)\emptyset_{c-1}(s)\right\}/s$

And

$$A = x_0(x_0-1)\{1+(\lambda-\in)\, P^*{}_{0,0}(t)\emptyset_{c-1}(s)\}/\{sx_0+c(\mu-\in)(x_0-1)\} \qquad \text{……..(1.4.11)}$$

Since the Laplace Transform of the sum of all the probabilities is $1/s$ at any time

$$A(x_0-1)+\sum_{r=0}^{c-1} P^*{}_{r,0}(t)=\frac{1}{s} \qquad \text{……..(1.4.12)}$$

Substituting equations (1.4.11) and (1.4.8) in equation (1.4.12) , we get

$$x_0\{1+(\lambda-\mathrm{c}\in)\, P^*{}_{0,0}(t)\emptyset_{c-1}(s)\}/\{sx_0+c(\mu-\in)(x_0-1)\}$$

$$+P^*{}_{0,0}(t)\sum_{r=0}^{c-1}\emptyset_r(s)=\frac{1}{s} \qquad \text{……..(1.4.13)}$$

Hence,

$$P^*{}_{0,0}(t)=c(\mu-\in)(x_0-1)/s\{(\lambda-\mathrm{c}\in)x_0\emptyset_{c-1}(s)+[sx_0+c(\mu-\in)(x_0-1)]\sum_{r=0}^{c-1}\emptyset_r(s)\} \qquad \text{……..(1.4.14)}$$

From equations (1.4.14) , (1.4.11) and (1.4.8) the Laplace Transform of the state probabilities

can be obtained explicitly as ,

$$P^*{}_{c,n}(t)=-x^{-n}(x_0-1)\{(\lambda-\mathrm{c}\in)(s)+s\sum_{r=0}^{c-1}\emptyset_r(s)\}/s\{(\lambda-\mathrm{c}\in)x_0\emptyset_{c-1}(s)+[sx_0+\ c(\mu-\in)(x_0-1)]\sum_{r=0}^{c-1}\emptyset_r(s)\}$$

$$P^*{}_{m,0}(t)=\emptyset_m(s)c(\mu-\in)(x_0-1)/s\{(\lambda-\mathrm{c}\in)x_0\emptyset_{c-1}(s)+[sx_0+c(\mu-\in)(x_0-1)]\sum_{r=0}^{c-1}\emptyset_r(s)\} \qquad \text{……..(1.4.15)}$$

We obtain the steady state probabilities by taking the limit as $s\to 0$ of the product of s and the corresponding Laplace Transform. Thus, we have

$$P_{c,n}=P_{0,0}\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}\left(\frac{1}{k^n}\right) \quad for\ n>c \qquad \text{……..(1.4.16)}$$

$$P_{m,0}=P_{0,0}\left\{\prod_{i=1}^{m}\left[\frac{\lambda-\mathrm{i}\in}{c(\mu-\in)}\right]\right\} \quad for\ m>c \qquad \text{……..(1.4.17)}$$

Using the boundary condition, we have

$$P_{0,0}=\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\left(\frac{1}{c!}\right)(1-k)^{-1}+\sum_{m=0}^{c-1}\left[\prod_{m=0}^{c-1}\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}^{-1} \qquad \text{……..(1.4.18)}$$

Where k is the solution of the equation, we have

$$(\lambda-\mathrm{c}\in)(k-1)=c(\mu-\in)(k^b-1) \qquad \text{……..(1.4.19)}$$

The probability that the system is empty is

$$P_{0,0}=\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\left(\frac{1}{c!}\right)(1-k)^{-1}+1+\sum_{m=1}^{c-1}\left[\prod_{i=1}^{c-1}\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}^{-1} \qquad \text{……..(1.4.20)}$$

Where k is the solution of the equation (1.4.19)

For various values of c, ∈ and b and for fixed values of λ, μ the values of $P_{0,0}$ are computed and presented in Table (1.1) .From table (1.1) and equation (1.4.20) it is observed that P is increasing as the dependence parameter increases positively when other parameter are fixed . The values of $P_{0,0}$ increased as the batch size increases for fixed values of the other parameters. When other parameters are fixed the $P_{0,0}$ values is increasing as the number of servers increases. It is also further observed that as the arrival rate increases the $P_{0,0}$ values is decreasing and the service rate increases the $P_{0,0}$ values is increasing for fixed values of the remaining parameters.

The probability that the server is empty can be obtained as,

$$1-\sum_{n=0}^{\infty}P_{c,n}$$

$$1-\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\left(\frac{1}{c!}\right)(1-k)^{-1}+1+\sum_{m=1}^{c-1}\left[\prod_{i=1}^{m}\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}^{-1}$$

$$\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}\left(\frac{1}{k^n}\right) \qquad \text{……..(1.4.21)}$$

Where k is the solution of the equation (1.4.19)

The average number of busy servers is given by

$$\sum_{m=0}^{c-1} mP_{m,0} + \sum_{n=0}^{\infty} CP_{c,n}$$

$$\left[\sum_{m=1}^{c-1} m\left[\prod_{i=1}^{m}\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right] + c\sum_{n=0}^{\infty}\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}\left(\frac{1}{k^n}\right)\right.$$

$$\left.\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\left(\frac{1}{c!}\right)(1-k)^{-1} + 1 + \sum_{m=1}^{c-1}\left[\prod_{i=1}^{m}\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}^{-1}\right] \qquad \text{……..(1.4.22)}$$

Where k is the solution of the equation (1.4.19)

The average number of customers in the queue can be obtained as,

$$L_q = \sum_{n=1}^{\infty} mP_{c,n}$$

$$=\sum_{n=1}^{\infty} n\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}\left(\frac{1}{c!}\right)P_{0,0}$$

$$=\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}P_{0,0}\frac{k}{(1-k)^2}$$

$$=\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}\left\{\prod_{i=1}^{c}\left[\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\left(\frac{1}{c!}\right)(1-k)^{-1} + 1 + \sum_{m=1}^{c-1}\left[\prod_{i=1}^{m}\frac{\lambda-\mathrm{i}\in}{i(\mu-\in)}\right]\right\}^{-1}\frac{k}{(1-k)^2} \qquad \text{……..(1.4.23)}$$

Where k is the solution of the equation (1.4.19)

For various values of c, $\in$ and b and for fixed values of λ, μ the values of L_q are computed and are given in Table (1.3) .From table (1.3) and equation (1.4.23) we observe that L_q is a decreasing function of the dependence parameter $\in$. Similarly, The mean number of customers in the queue is decreasing as the batch size increases and this increase is more rapid in the initial values than the later values .The L_q is a decreasing function of C.

Table (1.2)
VALUES OF P_0
$\lambda = 4$ and $\mu = 6$

$\in/b$	1	2	3
C=1			
0	0.3331	0.4486	0.4658
0.4	0.3751	0.4636	0.4793
0.8	0.3849	0.4808	0.4961
C=2			
0	0.5001	0.5081	0.5092
0.4	0.5269	0.5313	0.5320
0.8	0.5555	0.5582	0.5585
C=3			
0	0.6632	0.6643	0.6645
0.4	0.7122	0.7125	0.7127
0.8	0.7690	0.7691	0.7692

Table (1.3)
VALUES OF P_0
$\lambda = 4$ And $\mu = 6$

$\in/b$	1	2	3
C=1			
0	1.3357	0.0832	0.0120
0.4	1.1580	0.0523	0.0053
0.8	0.9826	0.0308	0.0014
C=2			
0	0.4652	0.0548	0.0093
0.4	0.4293	0.0384	0.0044
0.8	0.3918	0.0236	0.0005
C=3			
0	0.3849	0.0512	0.0089
0.4	0.3593	0.0365	0.0043
0.8	0.3279	0.0028	0.0013

V. CONCLUTION: We observe that by regulating any one of these three parameters named, mean dependence rate, batch size (service capacity), number of service facilities we can reduce the mean queue length and hence the mean delay. This feature has a potential importance in developing the optimal operating policies .This model also includes the $M/M/1$ queuing model, $M/M/c$ interdependent queuing model, $M/M/1$ interdependent queuing model with bulk service and $M/M/c$ with bulk service model as particular cases . So, this model can be viewed as a generalized Poisson queuing model with bulk service.

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