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# The number of $i^{th}$ smallest parts of $r-partitions$ of $n$

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**Abstract:** George E Andrews [1] derived generating function for the number of smallest parts of *partitions* of positive integer *n*. Hanuma Reddy [2] defined $i^{th}$ smallest part and derived a relation between the $i^{th}$ smallest and $i^{th}$ greatest parts of *partitions* of *n* in general form. Here we derive generating function for the number of the $i^{th}$ smallest parts of $r-partitions$ of *n.*

**Keywords:** *partitions*, *r-partitions*, smallest parts of *partition* and $i^{th}$ smallest parts of *partition* of positive integer *n.*

**Subject classification:** 11P81 Elementary theory of *partitions*.

**Introduction**:

A *partition* of a positive integer $n$ is a finite non increasing sequence of positive integers $\lambda_1,\lambda_2,...,\lambda_r$ such that $\sum_{i=1}^{r}\lambda_i=n$ and is denoted by $n=(\lambda_1,\lambda_2,...,\lambda_r)$, $n=\lambda_1+\lambda_2+\lambda_3+...\lambda_r$ or $\lambda=(\lambda_1^{f_1},\lambda_2^{f_2},\lambda_3^{f_3},...)$ when $\lambda_1$ repeats $f_1$ times, $\lambda_2$ repeats $f_2$ times and so on. The $\lambda_i$ are called the parts of the *partition*. In what follows $\lambda$ stands for a *partition*of $n$, $\lambda=(\lambda_1,\lambda_2,...,\lambda_r),\lambda_1\ge\lambda_2\ge...\ge\lambda_r$. The set of all *partitions* of $n$ is represented by $\xi(n)$ by and its cardinolity $p(n)$.

If$1\le r\le n$ then $\xi_r(n)$ is the set of *partitions* of *n* with *r* parts and its cardinality is denoted by $p_r(n)$. A *partition* of *n* with exactly *r* parts is called $r-partition$ of *n*. We define

$$p_r(n)=\begin{cases}0 & \text{if} \quad r=0 \quad or \quad r>n\\ \text{number of } r-partitions \text{ of } n & \text{if} \quad 0<r\le n\end{cases}$$

$spt(n)$ denotes the number of smallest parts including repetitions in all *partitions* of $n$. $spt_i(n)$ denotes the number of $i^{th}$ smallest parts including repetitions in all *partitions* of $n$. $r-spt_i(n)$ denotes the number of $i^{th}$ smallest parts in all $r-partitions$ of *n*.The number of *partitions* of $n$ with least part greater than or equal to $k$ is represented by $p(k,n)$.

**1.1** Existing generating functions are given below.

| Function | Generating function | |
|---|---|---|
| $p_r(n)$ | $\dfrac{q^r}{(q)_r}$ | |
| $p_r(n-k)$ | $\dfrac{q^{r+k}}{(q)_r}$ | |
| number of divisors | $\displaystyle\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)}$ | |
| sum of divisors | $\displaystyle\sum_{n=1}^{\infty}\frac{n.q^n}{(1-q^n)}$ | (1.1.1) |

where $(q)_k=\prod_{n=1}^{k}(1-q^n)$ for $k>0$, $(q)_k=1$ for $k=0$ and $(q)_k=0$ for $k<0$.

and $(a)_n=(a;q)_n=(1-a)(1-aq)(1-aq^2)...(1-aq^{n-1})$

**1.2Theorem:**If $k\in N$ and $1\le k\le\left[\frac{n}{r}\right]$, then the number $f_r^{\,i}(k,n)$ of $r-partitions$ of $n$ with $k$ as $i^{th}$ smallest part is

i) $f_r(k,n)=p_{r-1}\left[n-(k-1)r-1\right]+\beta$ for $i=1$

where $\beta=\begin{cases}1 & \text{if } \dfrac{n}{r}=k\\ 0 & \text{otherwise}\end{cases}$

$ii)$ If $i>1$

$$f_r^{\,i}(k,n)=\sum_{r_1=1}^{r-1}\sum_{\mu_l=1}^{\infty}...\sum_{r_{i-2}=1}^{r_{i-3}-1}\sum_{\mu_{l-i+2}=1}^{\infty}p_{r_{i-1}-1}\begin{bmatrix}(n-r\mu_l-r_1\mu_{l-1}-...-r_{i-2}\mu_{l-i+2})\\ -(k-1)(r-\alpha_l...-\alpha_{l-i+2})-1\end{bmatrix}$$

$$+\sum_{r_1=1}^{r-1}\sum_{\mu_l=1}^{\infty}...\sum_{r_{i-2}=1}^{r_{i-3}-1}\sum_{\mu_{l-i+2}=1}^{\infty}\beta_{i-1} \qquad \text{for } i>1$$

where $\beta_{i-1}=\begin{cases}1 & \text{if } \dfrac{n-r\mu_l-r_1\mu_{l-1}-...-r_{i-2}\mu_{l-i+2}}{r_{i-1}}=k-\mu_l...-\mu_{l-i+2}\\ 0 & \text{otherwise}\end{cases}$

**Proof :**

(i) For $i=1$

Let $n=(\lambda_1,\lambda_2,\dots,\lambda_r)=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},,\dots,\mu_{l-1}^{\alpha_{l-1}},k^{\alpha_l}\right)$ be any $r-partition$ of $n$ with $l$ distinct parts.

Put $t=1$ in theorem 1.2 in [3], we get the number of $r-partitions$ of $n$ with $k$ as smallest part is

$$f_r(k,n)=p_{r-1}(k,n-k)+\beta$$

$$\text{where } \beta=\begin{cases}1 & \text{if } \frac{n}{r}=k\\ 0 & \text{otherwise}\end{cases}$$

First replace $k+1$ by $k,r$ by $r-1,$ then replace $n$ by $n-k$ in theorem 1.3 in [3], we get

$$=p_{r-1}\left[n-(k-1)r-1\right]+\beta \quad (1.2.1)$$

(ii) For $i>1$

Let $n=(\lambda_1,\lambda_2,\dots,\lambda_r)$

$$=\left(\mu_1^{\alpha_1},\dots,\mu_{l-i}^{\alpha_{l-i}},\mu_{l-i+1}^{\alpha_{l-i+1}},\mu_{l-i+2}^{\alpha_{l-i+2}},\dots,\mu_{l-1}^{\alpha_{l-1}},\mu_l^{\alpha_l}\right) \quad (1.2.2)$$

be any $r-partition$ of $n$ with $l$ distinct parts. Subtracting $\mu_l$ from $\lambda_i$ for $i=1$ to $r$, we get

$$n_1=(\lambda_1^{(1)},\lambda_2^{(1)},\dots,\lambda_{r_1}^{(1)})=\left(\left(\mu_1^{(1)}\right)^{\alpha_1},\dots,\left(\mu_{l-i}^{(1)}\right)^{\alpha_{l-i}},\left(\mu_{l-i+1}^{(1)}\right)^{\alpha_{l-i+1}},\left(\mu_{l-i+2}^{(1)}\right)^{\alpha_{l-i+2}},\dots,\left(\mu_{l-1}^{(1)}\right)^{\alpha_{l-1}}\right)$$

$$\text{where } n_1=n-r\mu_l, r_1=r-\alpha_l \text{ and } \mu_\varphi^{(1)}=\mu_\varphi-\mu_l\forall\varphi \quad (1.2.3)$$

From (1.2.1) we have the number of $r_1-partitions$ of $n_1$ having smallest element $k$ is

$$p_{r_1-1}\left[n_1-(k-1)r_1-1\right]+\beta_1$$

$$\text{where } \beta_1=\begin{cases}1 & \text{if } \frac{n_1}{r_1}=k\\ 0 & \text{otherwise}\end{cases}$$

$$=p_{r_1-1}\left[(n-r\mu_l)-(k-1)r_1-1\right]+\beta_1 (1.2.4)$$

$$\text{where } \beta_1=\begin{cases}1 & \text{if } \frac{n-r\mu_l}{r_1}=k-\mu_l\\ 0 & \text{otherwise}\end{cases}$$

In (1.2.2), the part $\mu_l$ may vary from 1 to $\mu_{l-1}-1$ and $r_1$ may vary from 1 to $r-1$(if $\mu_l=\mu_{l-1}$ or $r_1=r,$ the *partition*(1.2.2) does not have $l$ distinct parts.

It contradicts our assumption for $\mu_l>\mu_{l-1}$.)

Therefore the number of $r-partitions$ of $n$ with second smallest part $k$ is $f_r^{\,2}(k,n)$

$$f_r^{\,2}(k,n)=\sum_{r_1=1}^{r-1}\sum_{\mu_l=1}^{\infty}p_{r_1-1}\left[(n-r\mu_l)-(k-1)r_1-1\right]+\sum_{r_1=1}^{r-1}\sum_{\mu_l=1}^{\infty}\beta_1 \quad (1.2.5)$$

Continuing this process in(1.2.3), we get

$$n_h = (\lambda_1^{(h)}, \lambda_2^{(h)}, \dots, \lambda_{r_h}^{(h)}) = \left(\left(\mu_1^{(h)}\right)^{\alpha_1}, \left(\mu_2^{(h)}\right)^{\alpha_2}, \dots, \left(\mu_{l-h-1}^{(h)}\right)^{\alpha_{l-h-1}}, \left(\mu_{l-h}^{(h)}\right)^{\alpha_{l-h}}\right)$$

where $n_0 = n, n_h = n_{h-1} - r_{h-1}\mu_{l-h+1}, r_0 = r,\ r_h = r_{h-1} - \alpha_{l-h+1}$ and $\mu_\varphi^{(h)} = \mu_\varphi^{(h-1)} - \mu_{l-h+1} \forall \varphi$

From (1.2.1), we have the number of $r_h - partition$ of $n_h$ having smallest part $k$ is

$$p_{r_h-1}\left[n_h - (k-1)r_h - 1\right] + \beta_h$$

$$\text{where } \beta_h = \begin{cases} 1 & \text{if } r_h \mid n_h \\ 0 & \text{otherwise} \end{cases}$$

Hence the number $f_r^i(k,n)$ of $r_{i-1} - partitions$ of $n_{i-1}$ with $i^{th}$ smallest part $k$ as

$$f_r^i(k,n) = p_{r_{i-1}-1}\left[n_{i-1} - (k-1)r_{i-1} - 1\right] + \beta_{i-1}$$

$$\text{where } \beta_{i-1} = \begin{cases} 1 & \text{if } r_{i-1} \mid n_{i-1} \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{r_1=1}^{r-1} \sum_{\mu_l=1}^{\infty} \dots \sum_{r_{i-2}=1}^{r_{i-3}-1} \sum_{\mu_{l-i+2}=1}^{\infty} p_{r_{i-1}-1}\left[\begin{matrix}(n - r\mu_l - r_1\mu_{l-1} - \dots - r_{i-2}\mu_{l-i+2}) \\ -(k-1)(r - \alpha_l \dots - \alpha_{l-i+2}) - 1\end{matrix}\right]$$

$$+ \sum_{r_1=1}^{r-1} \sum_{\mu_l=1}^{\infty} \dots \sum_{r_{i-2}=1}^{r_{i-3}-1} \sum_{\mu_{l-i+2}=1}^{\infty} \beta_{i-1}$$

$$\text{where } \beta_{i-1} = \begin{cases} 1 & \text{if } \dfrac{n - r\mu_l - r_1\mu_{l-1} - \dots - r_{i-2}\mu_{l-i+2}}{r_{i-1}} = k - \mu_l \dots - \mu_{l-i+2} \\ 0 & \text{otherwise} \end{cases}$$

This completes the proof ■

**1.3Theorem:**The generating function for the number of $i^{th}$ smallest parts of $r - partitions$ of $n$ such that $i^{th}$ smallest part as first part $\left(i.e\ \lambda_1 \text{ as } i^{th} \text{smallest part}\right)$ is

$$\sum_{n=1}^{\infty}\left(r - spt_i(n)\right)q^n = \frac{q^r}{(1-q^r)} \sum_{r_1=1}^{r-1} \frac{q^{r_1}}{(1-q^{r_1})} \sum_{r_2=1}^{r_1-1} \frac{q^{r_2}}{(1-q^{r_2})} \dots \sum_{r_{i-1}=1}^{r_{i-2}-1} \frac{q^{r_{i-1}}}{(1-q^{r_{i-1}})} \quad \text{for } i = r \quad (1.3.1)$$

**Proof:** Let $n = (\lambda_1, \lambda_2, \dots, \lambda_r) = (\mu^r)$ be any $r - partition$ of $n$ with all equal parts.

We know that $\beta$ is the number of smallest parts of $r - partitions$ of $n$ such that smallest part is the first part which is $k$ $\left(i.e\ \lambda_1 \text{ as smallest part}\right)$.

$$\text{where } \beta = \begin{cases} 1 & \text{if } \dfrac{n}{r} = k \\ 0 & \text{otherwise} \end{cases} \quad (1.3.2)$$

The generating function for the number of smallest parts of $r-partitions$ of $n$ such that smallest part is the first part $\left(i.e\ \lambda_1 \text{ as smallest part}\right)$ is

$$\sum_{n=1}^{\infty}\left(r-spt_1\left(n\right)\right)q^n=\sum_{k=1}^{\infty}q^{kr}=\frac{q^r}{\left(1-q^r\right)} \text{ for } r=1 \qquad (1.3.3)$$

Let $n=(\lambda_1,\lambda_2,\ldots,\lambda_r)=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2}\right)$ be any $r-partition$ of $n$ with two distinct parts.

Subtracting $\mu_2$ from each $\lambda_i$ for $i=1$ to $r$, we get

$$n_1=\left(\mu_1^{(1)}\right)^{\alpha_1} \quad \text{where} \quad n_1=n-r\mu_2,\quad r_1=r-\alpha_2 \quad \text{and} \quad \mu_1^{(1)}=\mu_1-\mu_2$$

The number of smallest parts of $r_1-partitions$ of $n_1$ such that the smallest part is the first part and having $k$ as a smallest part is $\beta_1$

$$\text{where } \beta_1=\begin{cases}1 & \text{if } r_1\,|\,n_1\\ 0 & \text{otherwise}\end{cases}$$

Since $n=n_1+r\mu_2$ and $\mu_1=k-\mu_2$, the number of second smallest parts of $r-partitions$ of $n$ such that second smallest part is the first part and having $k$ as a smallest part is $\beta_1$

$$\text{where } \beta_1=\begin{cases}1 & \text{if } \dfrac{n-r\mu_2}{r_1}=k-\mu_2\\ 0 & \text{otherwise}\end{cases}$$

Therefore the generating function for the number of second smallest parts of $r-partitions$ of $n$ such that second smallest part is the first part $\left(i.e\ \lambda_1 \text{ as second smallest part}\right)$ is

$$\sum_{n=1}^{\infty}\left(r-spt_2\left(n\right)\right)q^n=\sum_{\mu_2=1}^{\infty}\sum_{r_1=1}^{r-1}\sum_{k-\mu_2=1}^{\infty}q^{\mu_2 r+\left(k-\mu_2\right)r_1}$$

$$=\sum_{\mu_2=1}^{\infty}\sum_{r_1=1}^{r-1}\sum_{\mu_1=1}^{\infty}q^{\mu_1 r_1+\mu_2 r}$$

$$=\sum_{\mu_2=1}^{\infty}q^{\mu_2 r}\sum_{\mu_1=1}^{\infty}\sum_{r_1=1}^{r-1}q^{\mu_1 r_1}$$

$$=\frac{q^r}{\left(1-q^r\right)}\sum_{r_1=1}^{r-1}\frac{q^{r_1}}{\left(1-q^{r_1}\right)} \text{ for } r=2 \quad (1.3.4)$$

Continuing this process, we getthe generating function for the number of $i^{th}$ smallest parts of $r-partitions$ of $n$ such that $i^{th}$ smallest part as first part $\left(i.e\ \lambda_1 \text{ as } i^{th}\text{smallest part}\right)$ is

$$\sum_{n=1}^{\infty}\left(r-spt_i\left(n\right)\right)q^n=\frac{q^r}{\left(1-q^r\right)}\sum_{r_1=1}^{r-1}\frac{q^{r_1}}{\left(1-q^{r_1}\right)}\sum_{r_2=1}^{r_1-1}\frac{q^{r_2}}{\left(1-q^{r_2}\right)}\cdots\sum_{r_{i-1}=1}^{r_{i-2}-1}\frac{q^{r_{i-1}}}{\left(1-q^{r_{i-1}}\right)} \text{ for } i=r \quad \blacksquare$$

**1.4Theorem:** The number of smallest parts of $r-partitions$ of $n$ having $k$ as a smallest part is

$$\sum_{i=0}^{\infty} p_{r-1-i}\left[n-(k-1)r-1-i\right]+\beta$$

$$\text{where } \beta=\begin{cases}1 & \text{if } r\mid n\\ 0 & \text{otherwise}\end{cases}$$

**Proof:**From (1.2.10), the number of $r-partitions$ of $n$ with the smallest part $k$ is

$$f_r(k,n)=p_{r-1}\left[n-(k-1)r-1\right]+\beta$$

Fix $k\in\{1,2,...,n\}$.For $1\le i\le r$ the number of $r-partitions$ of $n$ with the $(r-i)$ smallest parts each being $k$ is the number of $r-partitions$ of $n-(r-i)k$. Summing over $i=1$ to $r$ we get the total number of $r-partitions$ of $n$ with $k$ as the smallest parts .

This number $\sum_{i=0}^{\infty} p_{r-1-i}\left[n-(k-1)r-1-i\right]+\beta$ where $\beta=\begin{cases}1 & \text{if } r\mid n\\ 0 & \text{otherwise}\end{cases}$ ■

**1.5Theorem:**The generating function for the number of $i^{th}$ smallest parts of $r-partitions$ of $n$ is

$$\sum_{n=1}^{\infty} r-spt_i(n)q^n=\frac{q^n}{(1-q^n)}\sum_{r_1=1}^{r-1}\frac{q^{r_1}}{(1-q^{r_1})}\sum_{r_2=1}^{r_1-1}\frac{q^{r_2}}{(1-q^{r_2})}\cdots\sum_{r_{i-1}=1}^{r_{i-2}-1}\frac{q^{r_{i-1}}}{(1-q^{r_{i-1}})}\left(\sum_{r_i=1}^{r_{i-1}-1}\frac{1}{(q)_{r_i}}+1\right)$$

**Proof**: From theorem1.4, we have the number of smallest parts of $r-partitions$ of $n$ having $k$ as a smallest part is

$$\sum_{i=0}^{\infty} p_{r-1-i}\left[n-(k-1)r-1-i\right]+\beta \text{ where } \beta=\begin{cases}1 & \text{if } \frac{n}{r}=k\\ 0 & \text{otherwise}\end{cases}$$

The generating function for the number of smallest parts of $r-partitions$ of $n$ is

$$\sum_{n=1}^{\infty} r-spt(n)q^n=\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\frac{q^{r-1-i+(k-1)r+1+i}}{(q)_{r-1-i}}+\frac{q^r}{(1-q^r)}$$

$$=\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\frac{q^{kr}}{(q)_{r-1-i}}+\frac{q^r}{(1-q^r)}$$

$$=\sum_{i=0}^{\infty}\frac{(q^r+q^{2r}+q^{3r}+\ldots)}{(q)_{r-1-i}}+\frac{q^r}{(1-q^r)}$$

$$=\frac{q^r}{(1-q^r)}\sum_{i=1}^{r-1}\frac{1}{(q)_i}+\frac{q^r}{(1-q^r)}$$

$$=\frac{q^r}{\left(1-q^r\right)}\left(\sum_{i=1}^{r-1}\frac{1}{(q)_i}+1\right)$$

$$=\frac{q^r}{\left(1-q^r\right)}\left(\sum_{i=1}^{r-1}\frac{1}{(q)_i}+1\right)$$

$$=\frac{q^r}{\left(1-q^r\right)}\left(\sum_{r_1=1}^{r-1}\frac{1}{(q)_{r_1}}+1\right)$$

From theorem 1.4 and theorem1.6, we get the number of second smallest parts of $r-partitions$ of $n$ with second least part $k \neq \lambda_1$ is

$$f_r^{\,2}(k,n)=\sum_{r=1}^{\infty}\sum_{\mu_0=1}^{\infty}\sum_{r_1=1}^{r-1}\sum_{i=1}^{\infty}p_{r_1-1-i}\left[\left(n-r\mu_0\right)-\left(k-1\right)r_1-1-i\right]$$

The generating function for the number of second smallest parts $\neq \lambda_1$ of $r-partitions$ of $n$

$$\sum_{r=1}^{\infty}\sum_{\mu_0=1}^{\infty}\sum_{r_1=1}^{r-1}\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\frac{q^{r_1-1-i+r\mu_0+(k-1)r_1+1+i}}{(q)_{r_1-1-i}}$$

$$=\sum_{r=1}^{\infty}\sum_{\mu_0=1}^{\infty}\sum_{r_1=1}^{r-1}\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\frac{q^{r\mu_0+kr_1}}{(q)_{r_1-1-i}}$$

$$=\sum_{\mu_0=1}^{\infty}\sum_{r=1}^{\infty}q^{r\mu_0}\left[\sum_{r_1=1}^{r-1}\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\frac{q^{kr_1}}{(q)_{r_1-1-i}}\right]$$

$$=\frac{q^r}{\left(1-q^r\right)}\sum_{r_1=1}^{r-1}\frac{q^{r_1}}{\left(1-q^{r_1}\right)}\left(\sum_{r_2=1}^{r_1-1}\frac{1}{(q)_{r_2}}\right) \qquad (1.5.1)$$

From(1.3.4) the generating function for the number of second smallest parts of $r-partitions$ of $n$ with second smallest part equal to $\lambda_1$ is

$$\frac{q^r}{\left(1-q^r\right)}\sum_{r_1=1}^{r-1}\frac{q^{r_1}}{\left(1-q^{r_1}\right)} \quad \text{for } r=2 \qquad (1.5.2)$$

From (1.5.1) and (1.5.2) we get the generating function for the number of second smallest parts of $r-partitions$ of $n$ which is given by

$$\sum_{n=1}^{\infty}r-spt_2(n)q^n=\frac{q^r}{\left(1-q^r\right)}\sum_{r_1=1}^{r-1}\frac{q^{r_1}}{\left(1-q^{r_1}\right)}\left(\sum_{r_2=1}^{r_1-1}\frac{1}{(q)_{r_2}}\right)+\frac{q^r}{\left(1-q^r\right)}\sum_{r_1=1}^{r-1}\frac{q^{r_1}}{\left(1-q^{r_1}\right)}$$

$$=\frac{q^r}{\left(1-q^r\right)}\sum_{r_1=1}^{r-1}\frac{q^{r_1}}{\left(1-q^{r_1}\right)}\left(\sum_{r_2=1}^{r_1-1}\frac{1}{(q)_{r_2}}+1\right) \qquad (1.5.3)$$

By induction, the generating function for the number of $i^{th}$ smallest parts of $r-partitions$ of $n$ is

$$\sum_{n=1}^{\infty} r-spt_i(n)q^n = \frac{q^n}{(1-q^n)}\sum_{r_1=1}^{r-1}\frac{q^{r_1}}{(1-q^{r_1})}\sum_{r_2=1}^{r_1-1}\frac{q^{r_2}}{(1-q^{r_2})}\cdots\sum_{r_{i-1}=1}^{r_{i-2}-1}\frac{q^{r_{i-1}}}{(1-q^{r_{i-1}})}\left(\sum_{r_i=1}^{r_{i-1}-1}\frac{1}{(q)_{r_i}}+1\right) \blacksquare$$

**1.7 Corollary**:

The generating function for the number of $k's$ as smallest part of

$r-partitions$ of $n$ is

$$q^{kr}\left[\sum_{i=1}^{r-1}\frac{1}{(q)_i}+1\right]$$

**1.8 Corollary:**The generating function for the number of $k's$ as smallest parts of *partitions*of $n$ is

$$\sum_{r=1}^{\infty} q^{kr}\left[\sum_{i=1}^{r-1}\frac{1}{(q)_i}+1\right]$$

**1.9 Corollary**: The generating function for the number of $r-partitions$ of $n$ having $i$ distinct integers is

$$\sum_{r_1=1}^{\infty}\frac{q^{r_1}}{(1-q^{r_1})}\sum_{r_2=1}^{r_1-1}\frac{q^{r_2}}{(1-q^{r_2})}\cdots\sum_{r_i=1}^{r_{i-1}-1}\frac{q^{r_i}}{(1-q^{r_i})}$$

**1.10 Corollary:**The generating function for the number of smallest parts of $r-partitions$ of $n$ which are multiples of $c$ is

$$\sum_{n=1}^{\infty} c(r-spt_i(n))q^n = \frac{q^{cr}}{(1-q^{cr})}\left[\sum_{r_1=1}^{r-1}\frac{1}{(q)_{r_1}}+1\right]$$

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