Strong dominating sets and Strong Domination Polynomial of Complete Graphs

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Abstract

Let G = (V, E) be a simple graph. A set S $\subseteq$ V is called a dominating set if every vertex v $\in$ V is either a member of S or adjacent to a member of S. A set S $\subseteq$ V is a strong dominating set of G if for every u $\in$ V – S, there exists a v $\in$ S such that uv$\in$E and deg (u) $\leq$ deg (v). Let k_m be complete graph with order m. Let $Sd(K_m^j)$ be the family of strong dominating sets of a complete graph K_m with the number of elements in the set j, and let $Sd(K_m,j)$ = $|Sd(K_m^j)|$. In this paper, we establish K_m and obtain a iterative formula for $Sd(K_m,j)$. Using this iterative formula, we consider the polynomial SD (K_m, x) $= \sum_{j=1}^{m} Sd(K_{m,}\ j)\ x^j$, which we call strong domination polynomial of complete graphs and obtain some examples of this polynomial.

1 Introduction

Let G = (V, E) be a simple graph of order |V| = m. A set S $\subseteq$ V is called a dominating set if every vertex v $\in$ V is either a member of S or adjacent to a member of S. A set S $\subseteq$ V is a strong dominating set of G if for every u$\in$V–S, there v$\in$S such that uv$\in$E and deg (u) $\leq$ deg (v). The minimum cardinality of strong dominating set is called minimum strong domination number and is denoted by γ_{sd} (G). Alkhani and Peng found the dominating sets and domination polynomial of cycles and certain graphs [2], [3]. Abdul Jalil M. Khalaf and Sahib Shayyal Kahat found the dominating sets and domination polynomial of complete graph with missing edges [1]. Gehet, Khalf and Hasni found the dominating set and domination polynomial of stars and wheels [4] [5]. Let H_m be a graph with order m and let H_m^j be the family of dominating sets of a graph H_m with the number of elements in the set

j and let $d(H_{m,}\ j) = |H_m^j|$. We call the polynomial $D(H_{m,}\ x) = \sum_{j=\gamma(H)}^{n} d(H_{m,}\ j)\ x^j$, the domination polynomial of graph G [3]. Let K_m^j be the family of strong dominating sets of a complete graph K_m with the number of elements in the set j and let Sd ($K_{m,}$ j) = $|K_m^j|$. We call the polynomial

$$SD(K_m, x) = \sum_{j=\gamma_{Sd}(K_m)}^{\gamma n} Sd(K_{m,}j)\ x^j,$$

the strong domination polynomial of complete graph. In the next section we establish the families of strong dominating sets of K_m with the number of elements in the set j by the families of strong dominating sets of K_{m-1} with number of elements j and j – 1. We explore the strong domination polynomial of complete graphs in section 3.

As usual we use $\binom{n}{i}$ or nC_i for the combination n to i and we denote the set {1, 2, …, n} simply by [n], and we denote deg (u) to degree of the vertex u and let

$\Delta(G) = \max \{\deg (u) \mid \forall\, u \in V (G)\}$ and

$\delta (G) = \min \{\deg (u) \mid \forall\, u \in V (G)\}$

2. Strong Dominating sets of complete graphs

Let K_m, $m \geq 3$ be the complete graph with m vertices, $V[K_m] = [m]$ and $E (K_m) = \{(v, u) : \forall\, v, u \in V (K_m)\}$. Let K_m^j be the family of strong dominating sets of K_m with the number of elements j. We shall explore the strong dominating sets of complete graph. To prove our main results we need the following Lemmas.

Lemma 1 . The following properties hold for all graph G.

(i) $|H_m^m| = 1$

(ii) $|H_m^{m-1}| = 1$

(iii) $|H_m^j| = 0$ if $j > m$

(iv) $|H_m^o| = 0$

Proof . Let G = (V, E) be a simple graph of order m, then

(i) $H_m^m = \{H\}$. Therefore, $|H_m^m| = 1$. (ii) $H_m^{m-1} = \{H - u \mid \forall\, u \in H\}$, Therefore $|H_m^{m-1}| = m$

(iii) There does not exists $K \subseteq H$ such that $|V (K)| > |V (H)|$. Therefore, $|H_m^j| = 0$, if $j > m$.

(iv) There does not exists $K \subseteq H$ such that $|V (K)| = 0$, ϕ is not strong dominating set of H. Therefore $|Hm^0| = 0$.

Lemma 2 [4]. The following properties are hold by definition of combination $\left(\binom{n}{i} = \frac{n}{i!(n-i)!} \text{ (or) } nc_i = \frac{n!}{i!\,(n-i)!}\right)$ for all $n \in Z^+$.

(i) $\binom{n}{n} = 1$ (ii) $\binom{n}{n-1} = n$ (iii) $\binom{n}{1} = n$

(iv) $\binom{n}{0} = 1$ (v) $\binom{n}{i} = 0$ if $i > n$.

Theorem 1. Let K_m be complete graph with order m, then $Sd(K_m, j) = \binom{m}{j}, \forall\, m \in Z^+$ and $j = 1, 2, \ldots n$.

Proof . Let K_m be a complete graph, since every vertex $u \in K_m$ there exists a $v \in K_m$ such that $uv \in E$ and $\deg (u) \leq \deg (v)$ then every subset of K_m with the number of elements of the set j, $\forall\, 1 \leq j \leq m$ is strong dominating sets of K_m, therefore $Sd (K_m, j) = \binom{m}{j}$.

Theorem 2 . Let K_m be complete graph with order m, then Sd (K_m, j) = Sd (K_{m-1}, j) + Sd $(K_{m-1}, j-1)$ $\forall$ j > 1 m > 1.

Proof . We have Sd $(K_m, j) = \binom{m}{j}$. To prove $\binom{m}{j} = \binom{m-1}{j} + \binom{m-1}{j-1}$

We have $\binom{m}{j} = \dfrac{m!}{j!(m-j)!}$

Now, $\binom{m-1}{j} = \dfrac{(m-1)!}{j!(m-1-j)!}$

$$= \frac{(m-1)!}{j(j-1)!\,(m-1-j)!}$$

Now, $\binom{m-1}{j-1} = \dfrac{(m-1)!}{(j-1)!\,(m-1-j+1)!}$

$$\binom{m-1}{j-1} = \frac{(m-1)!}{(j-1)!\,(m-j)\,(m-j+1)!}$$

Now, $\binom{m-1}{j} + \binom{m-1}{j-1} = \dfrac{(m-1)!}{j(j-1)!\,(m-1-j)!} + \dfrac{(m-1)!}{(j-1)!\,(m-j)\,(m-j-1)!}$

$$= (m-1)!\left[\frac{1}{j(j-1)!\,(m-1-j)!} + \frac{1}{(j-1)!\,(m-j)\,(m-j-1)!}\right]$$

$$= (m-1)!\left[\frac{m-j+j}{j(j-1)!\,(m-j)\,(m-j-1)!}\right]$$

$$= \frac{m!}{j!\,(m-j)!}$$

$$= \binom{m}{j}$$

$$= Sd\,(K_m, j)$$

Therefore, Sd (K_{m-1}, j) + Sd $(K_{m-1}, j-1)$ = Sd (K_m, j).

Theorem 3 . The following characteristics hold for co efficient of SD (K_m, x), $\forall$ m $\in Z^+$.

(i) Sd $(K_m, 1)$ = m.

(ii) Sd (K_m, j) = Sd $(K_m, m - j)$.

(iii) If m is even number, then Sd $(Km, \frac{m}{2}+1)$ = Sd $(Km, \frac{m}{2}-1)$.

(iv) $\gamma_{Sd}(K_m) = 1$.

(v) Sd $(K_m, 2) = \dfrac{m(m-1)}{2}$ if $m \geq 2$.

Proof . Let K_m be a complete graph, then

(i) We have $Sd(K_m, j) = \binom{m}{j}$

$$Sd(K_m, 1) = \binom{m}{1}$$

$$= \frac{m!}{1!\,(m-1)!}$$

$$= \frac{m\,(m-1)!}{(m-1)!} = m$$

Therefore, Sd (K_m, 1) = m.

(ii) We have $\binom{m}{1} = \binom{m}{m-1}$

Let $Sd(K_m, j) = \frac{m!}{j!\,(m-j)!}$

Now, $Sd(K_m, m-j) = \frac{m!}{(m-j)!\,(m-m+j)!}$

$$= \frac{m!}{(m-j)!\,j!}$$

$$= Sd(K_m, j)$$

Therefore, Sd (K_m, j) = Sd (K_m, m-j).

(iii) If m is even number.

Now $Sd(Km, \frac{m}{2}+1) = Sd(K_m, \frac{m+2}{2}) = \binom{m}{\frac{m+2}{2}}$

$$= \frac{m!}{\left(\frac{m+2}{2}\right)!\left(m-\frac{m-2}{2}\right)!} = \frac{m!}{\left(\frac{m+2}{2}\right)!\left(\frac{m-2}{2}\right)!}$$

Now, $Sd\left(K_m, \frac{m}{2}-1\right) = Sd\left(K_m, \frac{m-2}{2}\right)$

$$= \binom{m}{\frac{m-2}{2}}$$

$$= \frac{m!}{\left(\frac{m-2}{2}\right)!\left(m-\frac{m+2}{2}\right)!}$$

$$= \frac{m!}{\left(\frac{m-2}{2}\right)!\left(\frac{m+2}{2}\right)!} = Sd\ (K_m, \frac{m}{2}+1)$$

Therefore, $Sd\left(K_m, \frac{m}{2}+1\right) = Sd\left(K_m, \frac{m}{2}-1\right)$, if m is even

(iv) Since {u}, ∀ u ∈ V (K_m) is a strong dominating set of (K_m), then γ_{Sd} (K_m) = 1.

(v) Now, Sd (K_m, 2) $= \binom{m}{2}$

$$= \frac{m!}{2!\,(m-2)!}$$

$$= \frac{m\,(m-1)}{2} \text{ if } m \geq 2.$$

Using Theorem 1 and Theorem 2, We obtain the coefficients of SD (K_m, x) for $1 \leq m \leq 20$ in Table 1. Let Sd (K_m, j) = $|K_m^j|$ There are interesting relationships between the numbers Sd (K_m,j), ($1 \leq j \leq n$) in the table.

m \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1																			
2	2	1																		
3	3	3	1																	
4	4	6	4	1																
5	5	10	10	5	1															
6	6	15	20	15	6	1														
7	7	21	35	35	21	7	1													
8	8	28	56	70	56	28	8	1												
9	9	36	84	126	126	84	36	9	1											
10	10	45	120	210	252	210	120	45	10	1										
11	11	55	165	330	462	462	330	165	55	11	1									
12	12	66	220	495	792	924	792	495	220	66	12	1								
13	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1							
14	14	91	364	1001	2002	3003	3532	3003	2002	1001	364	91	14	1						
15	15	105	455	1365	3003	5005	6535	6535	5005	3003	1365	455	105	15	1					
16	16	120	560	1820	4368	8008	11540	13070	11540	8008	4368	1820	560	120	16	1				
17	17	136	680	2380	6188	12376	19548	24610	24610	19548	12376	6188	2380	680	136	17	1			
18	18	153	816	3060	8568	18564	31924	44158	49220	44158	31924	18564	8568	3060	816	153	18	1		
19	19	171	969	3876	11628	27132	50488	76082	93378	93378	76082	50488	27132	11628	3876	969	171	19	1	
20	20	190	1140	4845	15504	38760	77620	126570	169460	186756	169460	126570	77620	38760	15504	4845	1140	190	20	1

Table 1 ($1 \leq m \leq 20$)

3. Strong Domination Polynomial of a complete Graphs

In this section we introduce and establish the strong domination polynomial of complete graphs.

Let K_m^j be the family of strong dominating sets of a complete K_m with cardinality j, and let $Sd(K_m, j) = |K_m^j|$ and since γ_{sd} (K_m) = 1. Then the strong domination polynomial SD (K_m, x) of K_m is defined as SD (K_m, x) = $\sum_{j=\gamma_{sd}(K_m)}^{na} Sd(K_m, j)\, x^j$.

Theorem 4 . The following characteristics hold for all SD $K_m, x)$, $\forall$ $m \geq 3$. (i) SD (K_m, x) = SD (K_{m-1}, x) + x SD (K_{m-1}, x) + x . (ii) SD $(K_m, x) = \sum_{j=1}^{m} \binom{m}{j} x^j$.

Proof . (i) From definition of the strong domination polynomial and Theorem 2, we have

$$SD(K_m, x) = \sum_{j=1}^{m} Sd(K_m, j)\, x^j$$

$$= \sum_{j=1}^{m} \left[Sd(K_{m-1}, j) + Sd(K_{m-1}, j-1\right] x^j$$

$$SD\ (K_m, x) = \sum_{j=1}^{m} Sd(K_{m-1}, j)\, x^j + \sum_{j=1}^{m} Sd(K_{m-1}, j-1)\, x^j$$

We have Sd $(K_m, j) = 0$ if $j > n$ by Lemma 1.

Then $\sum_{j=1}^{m} Sd(K_{m-1}, j)\, x^j = \sum_{j=1}^{m-1} Sd(K_{m-1}, j)\, x^j$

$$= SD\ (K_{m-1}, x)$$

and we have $Sd(K_{m-1}, j-1) = \binom{m-1}{j-1}$

$$= \binom{m-1}{0} \text{ if } j = 1$$

and $\sum_{j=2}^{m} Sd(K_{m-1}, j-1)\, x^{j-1} = \sum_{j=2}^{m-1} Sd(K_{m-1}, j)\, x^j$

Then $\sum_{j=1}^{m} Sd(K_{m-1}, j-1)\, x^j = x \sum_{j=1}^{m} Sd(K_{m-1}, j-1)\, x^{j-1}$

$$= x \left[\sum_{j=1}^{m-1} Sd(K_{m-1}, j)\, x^j + 1\right]$$

$$= x\ [SD\ (K_{m-1}, x) + 1]$$

$$= X\ SD\ (K_{m-1}, x) + x$$

Therefore, SD (K_m, x) = SD (K_{m-1}, x) + X SD (K_{m-1}, x) + x.

(ii) We have $SD\ (K_m, x) = \sum_{j=1}^{m} Sd(K_m, j)\, x^j$

$$\sum_{j=1}^{m} \binom{m}{j} x^j, \text{ by Theorem1.}$$

Example 1 . Let K_6 be complete graph with order 6, then $\gamma_{sd}(K_6) = 1$ and $SD(K_6, x) = \sum_{j=1}^{6} \binom{6}{j} x^j$

$= 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$ (See. Fig. 1).

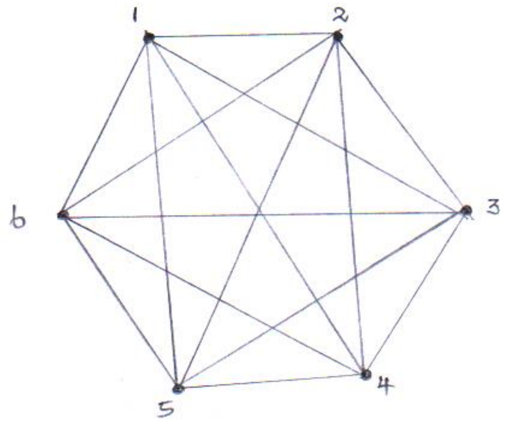

Figure 1. $H = K_6$ has $\binom{6}{j}$ Strong dominating set with cardinality i.

Example 2 . Let K_8 be complete graph with order 8, then $\gamma_{sd}(k_8) = 1$ and $SD(K_8, x) = \sum_{j=1}^{8} \binom{8}{j} x^j$

$= 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$ (See. Fig. 2)

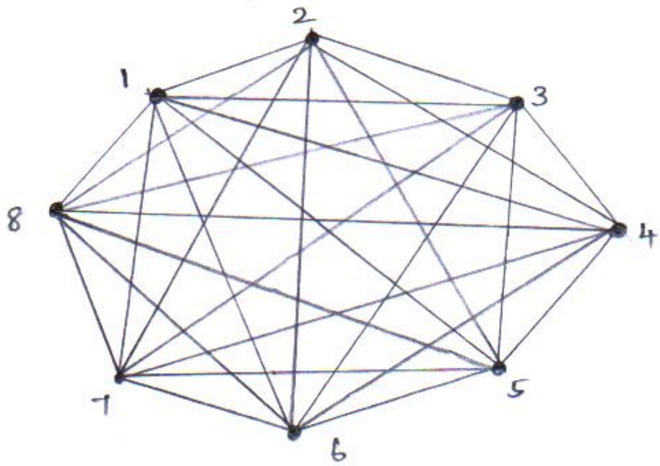

Figure 2 . K_8 has $\binom{8}{j}$ strong dominating set with cardinality j

4. References

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