FIRST ORDER DIFFERENTIAL EQUATION WITH CONSTANT DELAY

Abstract:

Differential equations play an important role in science, engineering and social sciences. They occur quite frequently in our daily life. The motion of an object can always be associated with a differential equation. The change in prices of commodities, the flow of liquids, the concentration of chemicals etc., often lead to differential equations.

Most of the equations occurring in applications, are not only depend on the current state but also depends on the past history and these types of equations are called delay differential equations.

I.Introduction:

Consider an analog of the delay differential equation,

$$\frac{dx}{dt} = f(t, x(t), x(t-\tau)) \tag{1.1}$$

where $\tau > 0$, $f : R^3 \to R$ and $t \geq t_0$. Integrating (1.1) from t_0 to t, we obtain

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s)x(s-\tau))ds. \tag{1.2}$$

To define a solution $x(t)$ of equation (1.2) in $[t_0, t_0+\tau]$, one needs to have a known function ϕ in $[t_0-\tau, t_0]$ such that $x(t) = \phi(t)$ for $t \in [t_0-\tau, t_0]$ instead of just at t_0, that is $x(t_0) = x_0$. Using method of steps, the solution $x(t)$ can be extended to any interval $[t_0, T]$, $T \geq t_0$. Here the point t_0 is known as initial point, the set $E_{(t_0)} = [t_0-\tau, t_0]$ is called a initial set and ϕ is called the initial function.

Under general assumptions of f, the existence and uniqueness of the initial value problem (1.1) are established, see for example, Driver [3] and Hale [5].

A solution $x(t)$ of equation (1.1) is said to be continuable if it exists in a half line $[T, \infty)$ for some $T \geq 0$. Throughout the paper we deal with continuable solutions without further mention. A non-trivial continuous function $x : [0, \infty) \to R$ is said to be oscillatory, if it has zeros for large t. That is, the set of zeros of $x(t)$ in any half interval $[T, \infty)$, $T \geq 0$ is unbounded. Similarly, $x(t)$ is said to be nonosciallatory, if it is not oscillatory. From the definition, it is clear that $x(t)$ is oscillatory if and only if, there exists a sequence of real numbers $\{t_n\}$ such that $x(t_n) = 0$ for every n and $t_n \to \infty$ as $n \to \infty$. Thus for a nonoscillatory function $x(t)$, there exists a constant $T_0 \geq 0$ such that $x(t)$ is either positive or negative for all $t \geq T_0$.

In oscillaton theory, it is always assumed that the solution of differential equation under consideration is non-trivial and continuable in a half line $[T, \infty)$, $T \geq 0$.

The distinguishing properties of a delay differential equation may be observed from that of an ordinary differential equation may be observed from the following example,

$$x'(t) + x(t-\pi/2) = 0. \tag{1.3}$$

This equation admits oscillatory solutions $x_1(t) = \sin t$ and $x_2(t) = \cos t$. However, it is a point to note that the corresponding ordinary differential equation,

$$x'(t) + x(t) = 0$$

admits solution of the form $x(t) = ce^{-t}$ which are non oscillatory. Similarly, if we consider a differential equation with advanced argument,

$$x'(t) - x(t+\pi/2) = 0, \tag{1.4}$$

we may see that $x_1(t) = \sin t$ and $x_2(t) = \cos t$ are oscillatory solutions and its associated ordinary differential equation,

$$x'(t) - x(t) = 0$$

has non oscillatory solution $x(t) = ce^{t}$ only. From these examples, it is understood that the oscillation of solutions in equations (1.3) and (1.4) are exclusively caused by the deviating argument $\pi/2$.

Although, it is not very transparent, it appears that Fite presented the first effective paper on oscillation of a class of functional differential equations of the form

$$x^{n}(t) + p(t)x(\tau(t)) = 0 \tag{1.5}$$

where $p{:}R \to R$ and $\tau(t) = k - t$ for some $k > 0$. His study aimed to find the behaviour of equation (1.5) which do not appear for the associated ordinary differential equation. The subject grow rapidly in the past several years, see for example, Wang, Stavroulakis and Qain [26] and the references cited therein.

The aim of this paper is to present briefly some results on the oscillatory and nonoscillatory behaviour of first order delay differential equations.In Chapter 2, we study the oscillatory behaviour of all solutions of the first order delay differential equations of the form

$$x'(t) + p(t)x(t-\tau) = 0, \qquad t \ge t_0$$

where $p(t)$ is positive and continuous functions, τ is a positive real constant.In Chapter 3, we discuss the oscillatory behaviour of solutions of the first order delay differential equation of the form

$$x'(t) + p(t)x(\tau(t)) = 0, \qquad t \ge t_0$$

where $p(t)$ is positive and continuous functions, $\tau(t) \in c([t_0, \infty), R^+)$, $\tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \infty$.Finally in Chapter 4, we discuss the oscillatory behaviour of solutions of the first order delay differential equation of the form

$$x'(t) + p(t)x(t - \tau(t)) = 0, \qquad t \ge t_0$$

where $p(t)$ is positive and continuous functions, $\tau(t) \in c([t_0, \infty), R^+)$, $\tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

2.1 Oscilatory Behaviour:

we study the oscillatory behaviour of all solutions of the delay differential equation of the form

$$x'(t) + p(t)x(t-\tau) = 0, \qquad t \ge t_0 \tag{2.1}$$

where $p(t)$ is a real valued continuous functions and τ is positive real number.

By a solution of equation (2.1), we mean a function $x(t)$ statisfies equation (2.1) for all $t \ge t_0$ and $x(t) = \phi(t)$ for all $t \in [t_0 - \tau, t_0]$. If the coefficient $p(t)$ in equation (2.1) is constant say p, then the condition $p\tau e > 1$ gives a necessary and sufficient conditions for all solutions of equation (2.1) to be oscillatory.

In this chapter, we present some oscillations results for the equation (2.1) when the function $p(t)$ is nonnegative. The results presented here are adopted from [9], [10] and [19] and references citied therein.

2.2.Oscillation Results: In this section we present some oscillation criteria for the equation (2.1) when τ is a positive constant and $p(t) \ge 0$ is a continuous function.

Theorem 2.1: Assume that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e} \tag{2.2}$$

and

$$\liminf_{t\to\infty} \int_{t-\tau/2}^{t} p(s)ds > 0. \tag{2.3}$$

Then every solution of equation (2.1) oscillates.

Proof : Suppose there is a solution $x(t)$ of equation (2.1) such that for t_0 sufficiently large $x(t)>0, t > t_0$.Then $x(t-\tau)$ > 0 for $t > t_0 + \tau$ and from equation (2.1), $x'(t) < 0$ for $t > t_0 + \tau$.Hence, $x(t) < x(t-\tau)$ for $t > t_0 + 2\tau$.

Set

$$w(t) = \frac{x(t-\tau)}{x(t)}, \qquad \text{for } t > t_0 + 2\tau. \tag{2.4}$$

Then $w(t) > 1$ and dividing both sides of equation (2.1) by $x(t)$, for $t > t_0 + 2\tau$ .We obtain,

$$\frac{x'(t)}{x(t)} + p(t)\frac{x(t-\tau)}{x(t)} = 0, \qquad t > t_0 + 2\tau.$$

$$\frac{x'(t)}{x(t)} + p(t)w(t) = 0, \qquad t > t_0 + 2\tau. \tag{2.5}$$

Integrating both sides of equation (2.5) from $t-\tau$ to t, for $t > t_0 + 3\tau$, yields

$$\ln w(t) = \int_{t-\tau}^{t} p(s)w(s)ds, \qquad t > t_0 + 3\tau. \tag{2.6}$$

Set

$$\liminf_{t\to\infty} w(t) = l. \tag{2.7}$$

Then $l \ge 1$ and is finite or infinite.

To prove this result it is sufficient to show that either case leads to a contradiction.

Case 1: l is finite.

Then there exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} w(t_n) = l$.

From equation (2.6), we find

$$\ln w(t_n) = \int_{t_n-\tau}^{t_n} p(s)w(s)ds$$

$$= w(\xi_n)\int_{t_n-\tau}^{t_n} p(s)ds \qquad (2.8)$$

where $t_n - \tau < \xi_n < t_n$, n=1,2, ...

Taking limits on both sides of equation (2.8), as $n \to \infty$, we obtain

$$\ln l \ge l \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds$$

and so

$$\frac{\ln l}{l} \ge \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds. \qquad (2.9)$$

Using the fact that,

$$\max_{l\ge 1} \frac{\ln l}{l} = \frac{1}{e}.$$

The inequality (2.9) implies

$$\frac{1}{e} \ge \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds$$

which contradicts the hypothesis (2.2).

Case 2 : l is infinite.

That is,

$$\liminf_{t\to\infty} \frac{x(t-\tau)}{x(t)} = \infty. \qquad (2.10)$$

Integrating both sides of equation (2.1) from $t-\tau/2$ to t, for $t > t_0 + 3\tau$, we obtain

$$x(t) - x(t-\tau/2) + \int_{t-\tau/2}^{t} p(s)x(s-\tau)ds = 0. \qquad (2.11)$$

Since $x(s-\tau) > x(t-\tau)$ for $t-\tau/2 \le s \le t$, equation (2.11) yields

$$x(t) - x(t-\tau/2) + x(t-\tau)\int_{t-\tau/2}^{t} p(s)ds \le 0. \qquad (2.12)$$

Dividing both sides of (2.12) by $x(t)$ and using (2.10) and (2.3) we conclude that

$$\lim_{t\to\infty} \frac{x(t-\tau/2)}{x(t)} = \infty. \qquad (2.13)$$

Dividing both sides of (2.12) by $x(t-\tau/2)$, we obtain

$$\frac{x(t)}{x(t-\tau/2)} - 1 + \frac{x(t-\tau)}{x(t-\tau/2)}\int_{t-\tau/2}^{t} p(s)ds \le 0 \qquad (2.14)$$

which in view of (2.13) and (2.3) is impossible. Since in both cases we arrived at a contradiction. Therefore, the proof of the Theorem is complete.

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