# A STUDY ON BANACH SPACE IN FUNCTIONAL ANALYSIS

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**Abstract:** What has category theory to offer to Banach spacers? In this survey-like paper we will focus on some of the five baselements of category theory –namely, i) The definition of category, functor and natural transformation; ii) Limits and co limits; iii) Adjoin functors; iv) The principles of categorical Banach space theory v) Banach space constructions as Banach functors.

**Key words**: Categorical Banach space theory, universal constructions, duality and adroitness

**Introduction:**

Functional analysis is important branch of mathematics research which is part of vector space and linear equations defined on spaces. In this properties of transformation of functional , fourier transformation, continuous functions, unitary …etc. in this study operations between the differential and integral.

In this general concept have been introduced by Italian mathematician and physicist vito volterra in 1887. The basic and historically first class of space studies in functional analysis is complex normed vector space over the real and complex numbers. In more important study of functional analysis is topological vector space and continuous linear vector space defined on banach space and Hilbert space.

**The principles of categorical Banach space theory**

A functor $F : A \to C$ between two categories is a correspondence assigning objects to objects and arrows to arrows which respects composition and identities. the functor is called covariant if whenever

$$f : A \to C \text{ then } F(f) : F(A) \to F(C).$$

Thus, what EM means, translated to the Banach space world is: I Banach space constructions must be understood and formulated as functions.

As it is formulated by Mityagin and Svarc [22]: ˇ "The essence of the matter is that nearly every natural construction of a new Banach space from a given Banach space generates a certain (covariant o contravariant) functor".

This is the foundational work. So let us start labelling Ban the category of Banach spaces, whose objects are Banach spaces and whose arrows are the linear continuous operators. It will be often necessary to work with the related category Ban1 whose objects are Banach spaces but whose arrows are only the linear continuous contractions.

• The identity covariant functor $\imath : B \to B$ which is defined in any subcategory B of Ban.

• The contravariant duality functor D defined by $D(X) = X^*$ for a Banach space X and $D(T) = T^*$ for an operator T.

• More generally, given a Banach space Y the contravariant L Y functor defined by L Y (X) = L(X, Y ) and L Y (T) = T ◦ with he meaning T ◦ (S) = ST. The choice Y = R gives the duality functor.

• Given a Banach space X the covariant LX functor defined by LX(Y ) = L(X, Y ) and LX(T) = T∗ with he meaning T∗(S) = T S. The choice X = R gives the identity.

• Given a Banach space X the covariant ⊗X functor defined by $\otimes X(Y) = X \otimes_{b\pi} Y$ and $\otimes X(T) = 1X \otimes T$.

• The covariant functors (see [17, 18]) assigning to a Banach space X the space $\ell p(X)$ of p-summable sequences with the natural induced operators.

• One can equally define the Grothendieck-Pietsch functors that assign to a Banach space X the space ℓ w p (X) of weakly p-summmable sequences on X.

**Banach space constructions as Banach functors:**

In this case, (a part of) what the EM says is that a correspondence lies on a categorical level only when it is a functor. When τ is a norm one operator then C(τ ) can be defined as C(τ )(f) = fτ ∗ . Thus, the correspondence establishes a functor when acting Ban1 → Ban1 –although Banach space tricks yield that to every operator τ : X → Y corresponds an "extension operator" T : C(BX∗ ) → C(BY ∗ ) given by ‖τ‖C ( τ ‖τ‖ ) . Of course, additivity has been lost.

On the covariant side, apart of the example previously considered, probably the simplest construction is that associating to a Banach space X the injective space l∞(BX∗ ). It was Semadeni [28] the first one to recognize a functor here. • X can be embedded into a separable L∞ Banach space L∞(X) in such a way that L∞(X)/X has the Radon-Nikodym and Schur properties.

**Banach envelopes**

Read in purely Banach space terms, the general situation is that every Banach space X can be naturally embedded into a space of continuous functions C(BX∗ ) in such a way that the embedding δX : X → C(BX∗ ) has the universal property that every C(K)-valued operator defined on X can be extended through δX to the whole C(BX∗ ).

The following subclasses of L∞-spaces have appeared in the literature:

1. Lindenstrauss spaces (denoted L); i.e., spaces that are L1+ε-spaces for all ε > 0.

2. Separably injective (Θ) and universally separably injective (Θu ) spaces.

3. Lindenstrauss-Pe lczy´nski spaces (LP). Recall from [85, 86] that a Banach space E is said to be a Lindenstrauss-Pe lczy´nski space if all operators from subspaces of c0 into E can be extended to c0. If some extension exists verifying ‖Tb‖ ≤ λ‖T‖ we shall say that E is an LPλ space.

4. L∞-spaces (L∞)

**Banach spaces as functors**

But there is more. The EM program considers that, even when a "single" construction is studied, it must be understood as a functor are made out of something, and thus the comprehension of the space is not "right", in the Eilenberg-MacLane sense, until the correspondence between the constituents and the final space has been clearly established as an understandable functor.

**Operators as natural transformations**

"A continuous functions" understanding spaces themselves as functors. Still to explain is the "embedding" part. To start with, the abstract point of view clearly demands that if one wants to construct a category with functors as objects, then a definition for arrows is required.

Definition of natural transformation

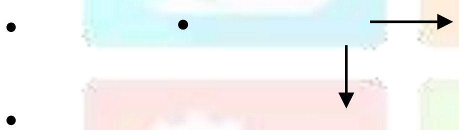
Natural transformation is arguably the most important notion in mathematics.Operators have to be understood as natural transformations. Banach spaces of natural transformations; The definition of Banach natural transformation is due to Mityagin and Svarc [22, 17, 18].

$$\|\tau\| = \sup X \|\tau X\|$$

is finite; here the supremum is taken over all the Banach spaces of the category B. Addition and multiplication by scalars can be defined in an obvious way for Banach natural transformations, and it is clear that the quantity above is a complete norm. So the space [F, G] of all Banach natural transformations between the Banach functors F and G is a Banach space.

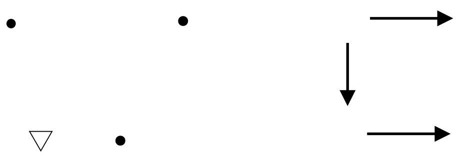
**Universal constructions: limits and colimits**

If Banach functor is the categorical way of saying "a correspondence that assigns to certain Banach spaces another Banach spaces", Universal construction is the categorical form of saying: limits and colimits. The prefix co- is the heart of categorical duality, to be considered later: whenever a statement (definition, theorem, . . . ) can be formulated in categorical terms, namely, in terms of points (objects) and arrows (i.e., diagrams) then there is a correspondent "dual" statement obtained reversing the arrows. To give an example, if given the situation.



there is a universal (i.e., well and uniquely defined) construction of an object ▽.

then the dual construction is the universal construction that allows one to pass from



It is a matter of choice which construction ▽ or △ will be called limit and which co-limit. But the natural choice should be that all constructions of one type will be called say limits and all belonging to the other will then be colimits.

Products,pull-backs, limits . . . are all of the same type; as well as co-products, push-outs, quotients, inductive limits. It would therefore be natural to call limits to the first and co-limits to the second. But Banach space tradition is illogical: indeed, tradition gave the names "inductive limit" to a construction of type ▽, and "product" to a construction of type △. The definition goes back to Grothendieck [11]. ;Let D defined a diagram with points and arrows. There is no difficulty in considering D as a category. A functor F : D → Ban means just drawing D with Banach spaces in the place of points and operators in the place of arrows. So, given a Banach functor F : D → Ban, a Banach space L(F) will be called the direct limit of F through D if there is a family of operators αd : F(d) → L(F), d ∈ D, making commutative the whole diagram in Ban in such a way that for every other Banach space X and family of morphisms βd with the same property, there is a unique operator α : L(F) → X making the whole diagram (i.e., αα d = βd) commutative. Inverse limit. Let D and F be as before. A Banach space L(F) will be called the inverse limit of F through D if there is a family of operators αd : L(F) → F(i), d ∈ D, making commutative the whole diagram in Ban in such a way that for every other Banach space X and family of morphisms βd with the same property.

**Classification of Banach**

There are much deeper results on the classification of Banach spaces which may be of use for applications. For example,

pace is homeomorphic to the Hilbert space l2 . This theorem shows that questions of a purely topological nature

Which arise in nonlinear analysis may often be reduced to questions in Hilbert space, a space in which it is much easier to perform certain kinds of constructions.

It should, however, be pointed out that to a large extent the theory of Banach spaces is at present carried out without having in mind specific applications to other areas of analysis. It is my feeling that Banach spaces (like finite groups for example) form an interesting and important object for an independent study. The fact that some simple looking and natural questions concerning Banach spaces turn out to be very difficult, a fact which in the past led some mathematicians away from Banach space theory, is to me one of the reasons to believe that the study of Banach spaces is important. The main problems in Banach space theory lead to some interesting questions in other areas of mathematics like topology, measure theory, probability, and in particular Fourier analysis and geometry (integral, combinatorial, and even differential) in finite-dimensional Euclidean spaces. Though Banach space theory contains some surprising (and perhaps at first glance discouraging) counterexamples, it contains also many interesting and difficult results which are more than just isolated results and certainly justify the use of the word theory. It is quite natural to assume that some of these theorems, once they are more widely known, will find applications outside the theory of Banach spaces.

In what follows the main emphasis will be on the study of the structure of Banach spaces. If we take the categorial standpoint such a study must be incomplete if we do not study at the same time the structure and classification problem of the relevant morphisms, i.e., of operators (usually linear) between Banach spaces. There is of course a vast literature on operator theory and in particular spectral theory. It seems, however, that the theory of operators on general Banach spaces is at present in a much less satisfactory form than the theory of Banach spaces themselves. Only in the case of Hilbert spaces is there at present a really deep and detailed theory of operators available. Thus, though Hilbert spaces form a special kind of Banach space the Hilbert space theory is "almost disjoint" from Banach space theory. The former deals with questions concerning operators which quite often look hopeless in the Banach space case, while the latter considers questions which are often trivial in the special case of Hilbert spaces.

We return to the main aim of this section, the study of classification of Banach spaces in terms of all or only a part of the structure which is endowed on such a space by its definition. It is of course trivial to study only the linear structure of a Banach space X. We single out now four nontrivial and different ways for looking at a Banach space.

1. As a metric space;

2. As a uniform space;

3. As a linear topological space;

4. As a Banach space, i.e., taking into account all the structure contained in the definition.

By looking at a Banach space as a metric space the natural morphisms we consider are of course the continuous functions, and two Banach spaces are identified if they are homeomorphic. When E is considered as a uniform space the natural morphisms are the uniformly continuous maps. In linear topological spaces the natural maps are the continuous linear operators and we look for invariants under isomorphism: bicontinuous one-to-one linear operators. Finally, in Case 4 the natural maps are linear norm decreasing operators and we look for invariants under isometries, i.e., norm preserving linear operators. There are also other natural ways to look at Banach spaces, for instance, for questions related to differential calculus on Banach spaces, but we will consider here only the four ways listed above.

A basic result in point set topology is that the dimension is a topological invariant of finite-dimensional Banach spaces. It is obviously the only topological (and even linear topological) invariant of these spaces. For infinite-dimensional separable Banach spaces Kadec's theorem gives a complete answer to the classification problem. For the nonseparable case the situation is not entirely clear. The following is conjectured.

By density character of a Banach space we mean the smallest cardinal for a dense subset of the space. The conjecture has been verified in some special classes of nonseparable spaces. For example, it is true if we restrict ourselves to reflexive Banach spaces (Bessaga [l]). Turning to the uniform classification, here, in contrast to the topological classification, the main open problem is whether uniformly homeomorphic Banach spaces are already isomorphic. That is, does the uniform structure of a Banach space already determine its linear (and thus linear topological) structure ?

1. If p >, max(2, q), an infinite-dimensional LJp) space is not uniformly homeomorphic to an infinite-dimensional L*(Y) space. This is probably always the case if p # q (i.e., also if p # q and both are smaller than two).
2. For infinite compact Hausdorff K, a C(K) space is not uniformly homeomorphic to a reflexive Banach space or to an Li(p) space.
3. 3. It is not known whether L,(O, 1) and Zr, are uniformly homeomorphic for somep, 1 < p < co, p # 2. Our guess is that they are not.

We come to the main subject of Banach space theory: The classification under isomorphisms and isometries. The isomorphic questions are usually the harder ones since we know much more isometric invariants. For example, the extremal structure of the unit ball of the space, or its dual form is a very useful isometric invariant which is usually useless for study of isomorphic questions. As an example of the difference between isomorphic and isometric classifications, let K be a compact metric space and consider C(K), the continuous functions on K with the supremum norm. Banach, in his monograph, proved thefollowing.

For compact metric H and K, H is homeomorphic to K if and only if C(H) is isometric to C(K).

The theorem follows immediately from the fact that the extreme points of the unit ball of C(K) * in its weak* topology (i.e., the topology induced by C(K)) can be (in the real case) identified as a disjoint union of two copies of K. What analog do we have for isomorphism ? Very simple examples show that C(K) may be isomorphic to C(H) without K being homeomorphic to H. It is however surprising (and not easy to prove) that even the dimension of H is not an isomorphic invariant of C(H). Milutin [9] proved the following

**THEOREM.**

If H and K are two uncountable compact metric spaces then C(H) is isomorphic to C(K).

The isomorphic classification of C(K) spaces with K countable compact metric spaces was done by Bessaga and Pelczynski [2]. There are uncountably many isomorphism classes of such spaces. The isomorphism invariant is obtained as follows: Let K', K",..., KcW),..., be the transfinite set of the derived sets of K. Since K is countable it does not have a nonempty perfect subset and thus K(O) = ,D for some countable 01. Let /3 be the smallest ordinal such that K(B) = O. The ordinal number /3~ (w the ordinal of the set of positive integers) is a linear topological invariant of C(K) which completely determines the isomorphism class of the C(K) s p ace (K compact metric countable).

Classification of C(K) spaces is far from being solved. (For some recent results concerning this question.We have already seen two examples of questions which have been solved in the separable case and are still open for non separable Banach spaces (viz., the questions of topological classification of Banach spaces and the linear topological classification of C(K) spaces). In general the study of non separable Banach spaces (though certainly not important from the point of view of applications) poses very interesting and hard questions. This study, which became only recently an area of systematic research, already produced some nontrivial results. One aspect of nonseparable Banach space theory will be discussed in the next lecture.

We make next some remarks on the problem of a structure theory for Banach spaces. At present there is no such theory available; we shall discuss here only a plausible candidate. It is quite clear that in any structure theory for Banach spaces the classical spaces, i.e., Lp(p) and C(K) spaces, must play a dominant role. This is true in particular for the classical sequence spaces c,, and II, , 1 < p < co.

**References:**


1. J.M.F. Castillo, Y. Moreno, The category of exact sequences ofBanach spaces, in "Methods in Banach Space Theory", Proceedings of theV Conference in Banach Spaces, C´aceres, 2004, (J.M.F. Castillo and W.B.Johnson eds.),London Math. Soc., Lecture Notes Ser. 337, Cambridge UniversityPress., Cambridge, 2006, 139 – 158.
2. A. Avil_es, F. Cabello, J.M.F. Castillo, M. Gonz_alez,Y. Moreno, On separably injective Banach spaces, preprint 2010.
3. S.A. Argyros, S. Mercourakis, Examples concerning heredityproblems of WCG Banach spaces, Proc. Amer. Math. Soc. 133 (3) (2005),773 – 785