Role of Growth Models in Studying Population Dynamics

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Abstract

Population dynamics is the branch of biology that studies the size and age composition of particular population, and also studies the rate of vital events such as birth rate, death rate, migration & immigration. While studying population growth models one has to study formulation, analysis and application of models that describe the dynamics of biological populations. By plotting scatter diagram of bivariate data by taking time on X axis & population on Y axis we get tentative idea of growth models. They are mainly Linear, Exponential, Logistic & Gompertz curve. The most popular model is Logistic in which population grows slowly & grows exponentially. An increase in population is expected to cause depletion of resources of all types. The rate of increase may be lowered in such situation & reaches its saturation point which is species specific level. This is also called as sigmoidal growth model.

Key Words: Population Dynamics, Growth Models, Linear, Exponential, Logistic, Gompertz

Introduction

Ecologists study the natural processes to determine the size and composition of plants & animals population. One can predict which population will stabilize or which get multiplied in stipulated time by studying population dynamics. Mathematical or Statistical ecology is a branch of mathematics which has history more than 210 years. Thomas Malthus's first principal was based on exponential law. He worked on modeling population size at various time periods which is known as Malthusian Growth Model. In nineteenth century Demographic studies were then carried out by Benjamin Gompertz & Pierre Francois Verhulst.

The population in statistical Ecology Management of renewable resources is an important topic. Today's numerous problems of environment have roots in human's impact on forests, depletion of natural resources. Because of heavy harvesting to compensate need of food grains, there is series of challenges before ecologists, economists, planners & administrators. Therefore there is need to develop statistical methodologies that may help to develop a better understanding of real life problem situation.

In the past 30 years, population dynamics has been complemented by deterministic mathematical form which is based on evolutionary biology. These studies are also useful in the study of infectious disease affecting populations. Various mathematical models of viral spread have been proposed and analyzed, and provide important results that can be applied to health policy decisions. The very basic equation of a population dynamic model is

Population change = Births − Deaths + Immigration – Emigration

Let Nt denote population size of a species at time point t. Nt is biomass of organism at time t or it is count of organisms at time t. and its abundance some time later, indicated by N (t + Δt). Where Δt is small interval of time. Making the balance equation more explicit, we can now write

N(t + Δt) − Nt = Number of births during Δt − Number of deaths during Δt

For sake of mathematical treatment we assume that Nt is a continuous quantity. It can be derived by dividing both sides of the equation by Δt

{N(t + Δt) − N(t) } / Δt =

{Number of births during Δt } / Δt – { Number of deaths during Δt }/ Δt

By taking the limit Δt → 0, the left-hand side of this equation becomes the derivative of N(t) with respect to time t, while the right-hand side becomes the difference between the rate with the rate at which Nt changes over time t is clearly a function Nt, thus

$$f(Nt) = \frac{\mathrm{dNt}}{\mathrm{dt}}$$

Change in the population is function of current population size. It may increase or decrease. What should be function f (model) is the question in deciding growth model. The different models in population dynamics stem out from the choice of function f . Let Nt denote population size at time t.

The simplest model is Linear Growth Model. Here it is considered that the differential equation is constant

$$\frac{dNt}{dt} = C \qquad \textit{Where} \quad C \textit{ is Constant}$$

Integrating above function with respect to **t** that is $\int d\,Nt = \int C\,dt$ gives

$Nt = C\ t + C_0$ where C_0 is constant of integration. To get its initial value we put t=0 so that

$N_0 = C_0$ which is known.

Therefore **$N_t = N_0 + C.\ t$** This equation is equation of line Y = a + b X

Where b is slope of line & a is y intercept. Here N_0 is intercept on Y axis (Population size) & C is the slope at which the population changes. If $C > 0$, population increases to infinity and if $C < 0$, population would decrease and eventually will be extinct and may be 0 . if C is equal to zero then population size remains at N_0 for all time points t (Constant).

For example if the population figures at time t is Nt & shown as per Figure 1 given below

Time t	0	1	2	3	4	5	6	7
Nt	6	9	13	15	19	21	24	28

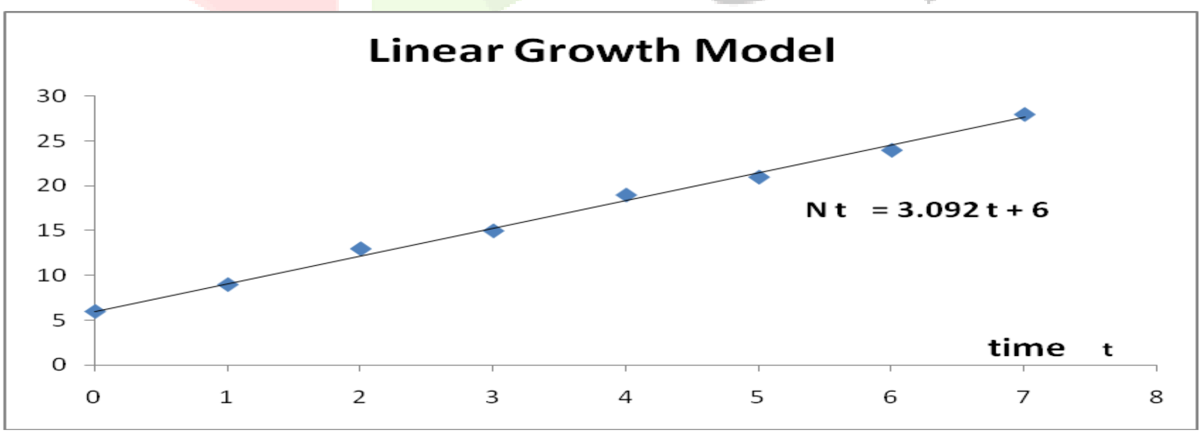

Figure 1- Linear growth model for given data

For the given data Mathematical Model is Linear **Nt= 3.092 * t + 6**

As t → Infinity then Nt→ Infinity

This Model is simple to understand but not realistic because the increment is constant over a time. This can be true for short interval of time but cannot true for longer period because population cannot be infinite at any given time. So one may consider another model where birth rate (b), death rate (m) is constant per animal, then change in population will be proportional to population size N. This can be written in mathematical form

$$\frac{dNt}{dt} = bNt - mNt = (b - m)Nt \;=\; k\,Nt\,.$$

Therefore $\frac{dNt}{dt}$ is proportional to Nt. This is second model which is called Exponential Model.

Exponential Model

When $\frac{dNt}{dt}$ is directly proportional to Nt. & k is constant of proportionality then k is called as intrinsic rate of increase or per capita instantaneous growth rate called as Malthusian parameter of the population under consideration. Therefore the net growth of the population is dependent on the size of the population. In this model population is increasing in an environment without upper limit & there is no interactions between the individuals of same species or other species within environment.

The exponential growth model as defined above is

$\frac{dNt}{dt} = k \;\; Nt$ which can be rewrite as $\frac{1}{Nt} * dNt - k\,dt = 0$

Integrating both sides with respect to t we have

$\int \frac{1}{Nt}\, d\,Nt = \int k\, dt$ Gives Log (Nt) = k t + C where C is constant of Integration.

To estimate initial value at t= 0 . $\{Nt\}_{at\ t=0} = N_0$ we get Log N_0 = C

Or $N_0 = Exp\,(\,C\,)$ Therefore Final mathematical Model is

Log N t = k t + Log N_0 OR

Exp(Log N t) = Exp(k t + Log N_0) = Exp(k t) * Exp(Log N_0)

Therefore Nt = N_0 * Exp (k t) is Exponential Model

If k > 0 There will be growth in population & If k <0 There will be decay at high rate.

For example consider a species having only two individuals in the beginning at t=0, then for k= 0.4

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Nt	2	3	4	7	10	15	22	33	49	73	109	163	243	363	541	807	1204	1796	2679	3996

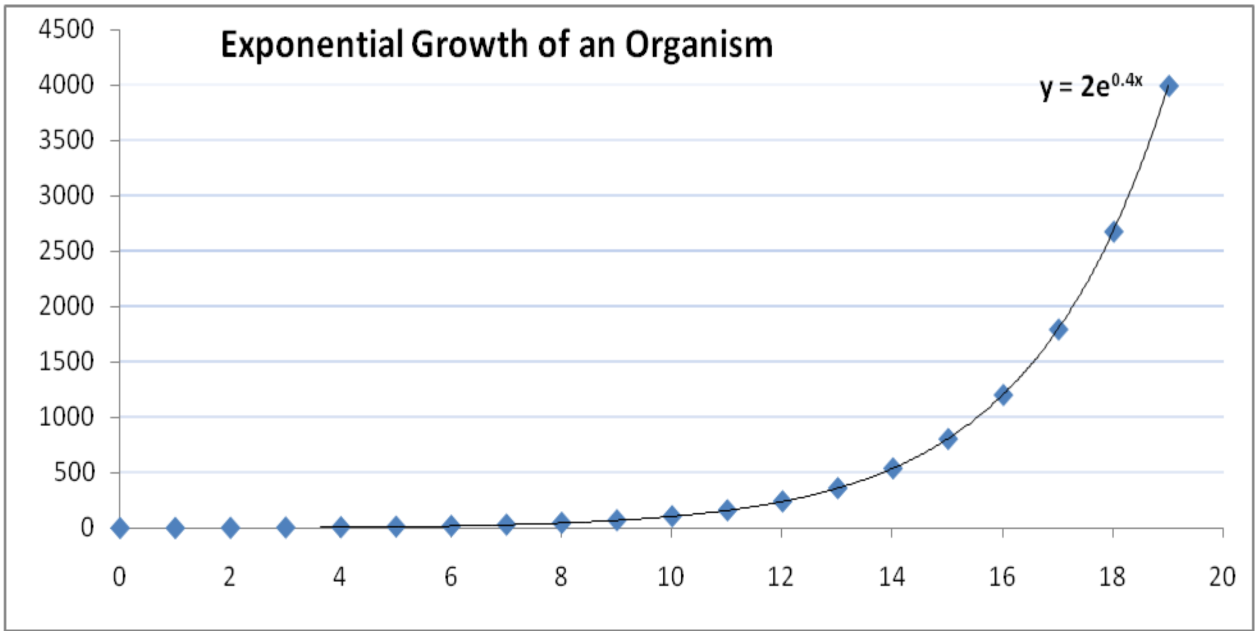

Figure 2- Exponential growth model for given data

These models are useful in modeling growth of bacteria, insects & paste (in a short time span)Exponential growth describes unregulated reproduction. Exponential growth is not a very sustainable state, since it depends on infinite amounts of resources which tend not to exist in the real world.

Logistic Growth Curve (Model)

A typical application of the logistic equation is a common model of population growth, originally due to Pierre-François Verhulst in 1838, where the rate of reproduction is proportional to both the existing population and the amount of available resources, all else being equal.

Exponential growth may happen for a while, if there are few individuals and many resources. But when the number of individuals gets large enough, resources start to get used up, slowing the growth rate. The availability of recourses in terms of food, space etc will in some manner arrest the population growth & total size of population. If the population density increases then the dearth rate of the population may also decrease sharply & further birth rate decreases.In due course of time, the growth rate will plateau, making an **S-shaped curve**.

The size at which it levels flat, which represents the maximum population size a particular environment can support, is called the **carrying capacity** or K.

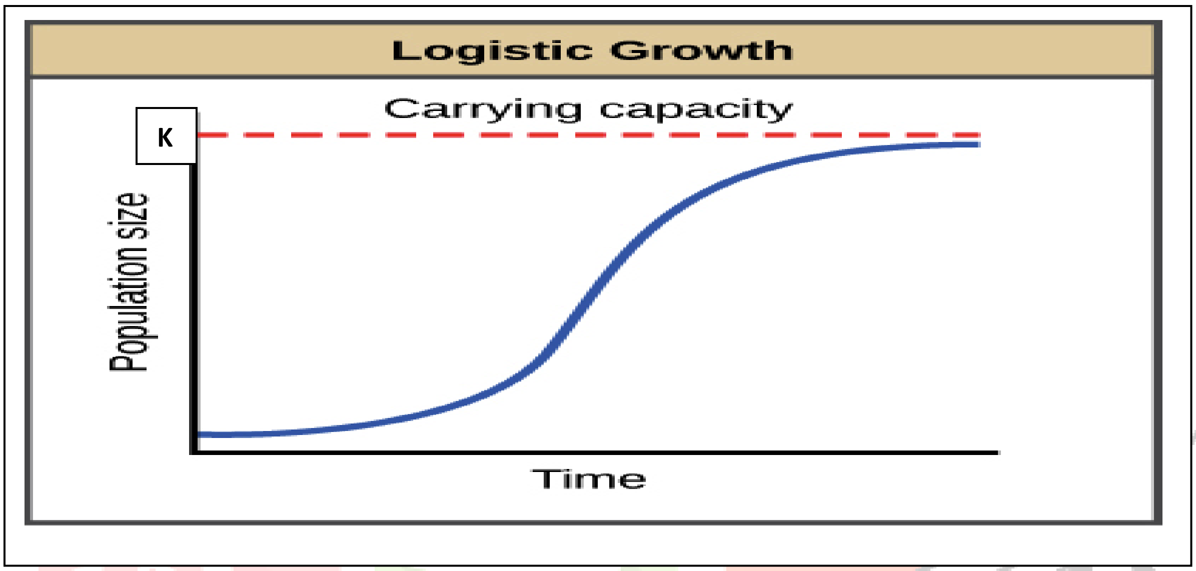

Figure 3- Logistic growth model

We can mathematically model logistic growth by modifying our equation for exponential

Growth, using per capita growth rate r that depends on population size Nt at time t and how close it is to maximum carrying capacity K which is also called as ceiling or equilibrium value. Assuming that the population has a base growth rate of r max, when it is very small, we can write the following equation:

$$\frac{dNt}{dT} = r\max \frac{(K - N)}{K} Nt$$

At any given point in time during a population's growth, the expression K-N , tells us how many more individuals can be added to the population before it hits carrying capacity. (K - N) / K then, is the fraction of the carrying capacity that has not yet been "used up." The more carrying capacity that has been used up, the more the (K - N)/K term will reduce the growth rate.

When the population is tiny, N is very small compared to K. The (K - N) / K term becomes approximately (*K*/*K*) or 1, giving us back the exponential equation. This fits with graph 3 above. The population grows near-exponentially at first, but levels off more and more as it approaches K. The Logistic model is obtained by integrating differential equation . Thus the final Logistic equation is

$$Nt = \frac{K}{1+Q\,Exp(-K\,r\,\max t)} \qquad Where\ Q = \frac{K-N0}{N0}$$

Conclusion:

Population growth rate measures the growth of individuals in size and length. Linear growth model for population & Exponential growth models describes unregulated reproduction. Population regulation is a density-dependent process, meaning that population growth rates are regulated by the density of a population. In such cases Logistic population growth models work more satisfactorily.

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