Semitopological Lattice Ordered Group

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Abstract
In this paper we make a study of semitopological lattice ordered (*lo*) group – a notion weaker than the well known one of topological lattice ordered groups. A topological lattice ordered group is always a semitopological lattice ordered *lo* group. But the converce is not true as shown by example. We derive here conditions that imply a semitopological *lo* group is a topological *lo* group.
Keywords : lattice ordered group, semitopological lattice, homeomorphisms.

Introduction
G. Birkhoff [1] defined a topological lattice ordered group with specified convergence in which the following hold :
(i) $x_\alpha \to x, y_\beta \to y \Rightarrow x_\alpha \wedge y_\beta \to x \wedge y$
(ii) $x_\alpha \to x, y_\beta \to y \Rightarrow x_\alpha \vee y_\beta \to x \vee y$

Semitopological lattice ordered group
Definition 1(a). A topological space that is also a lattice ordered group is called a semitopological lattice ordered (*lo*) group if the mapping
$g_1: (x, y) \to x \wedge y$
of $G \times G$ onto G is continuous in each variable separately.
(b) A topological space that is also a lattice ordered group is called a topological lattice ordered group if the mapping g_1 is continuous in both the variables together and if the inversion mapping $g_2 : x \to x^{-1}$ of G onto G is also continuous.
If the group operation is addition instead of multiplication, $x \wedge y$ and x^{-1} should be regarded as $x + y$ and $-x$ respectively. The identity of multiplicative group will be denoted by e that of an additive group by 0.
Preposition 1. Every topological lattice ordered group is a semitopological lattice ordered group. But the converse is not true.
Proof. The first statement is clearly true. To show that the converse is not true. Let $L = R$, the real line as an additive abelian group. Let L be endowed with a topology which has $\{[a, b) : -\infty < a < x < b < \infty\}$, the system of left closed and right open intervals as its base. Since for each neighbourhood $[a, b)$ of the identity $0, \left[a, \frac{b}{2}\right)$ is also neighbourhood of 0, it follows that the mapping is continuous in both variables together at 0. It is seen that g_1 is continuous everywhere. Hence G is a semitopological lattice ordered group. However, the mapping $g_2 : x \to -x$ is not continuous at 0 because if $[a, b)$ is a neighbourhood 0, then there is no neighbourhood V of 0 such that $-V \subseteq [0, b)$. Therefore, G is not a topological lattice ordered group. This completes the proof.

If we put $UV = \{xy : x \in U, y \in V\}$ and $U^{-1} = \{x^{-1}: x \in U\}$, where U and V are subset of L and in the additive case $U + V = \{x + y, x \in U, y \in V\}$, $-U = \{-x : x \in U\}$, then the continuity of the mappings g_1 and g_2 can be expressed as follow :

g_1 is continuous in x (or y) if, and only if, for each neighbourhood W of xy there exists a neighbourhood U (or V) of x (or y) such that $Uy \subseteq W$ (or $xV \subseteq W$). If L is abelian, then the right and left continuities $(x, y) \to xy$ in each variable are equivalent.

Moreover, g_1 is continuous is both x and y if, only if, for each neighbourhood W of xy there exists a neighbourhood U of x and a neighbourhood V of y such that $UV \subseteq W$. Similarly, g_2 is continuous if, and only if, for each neighbourhood W of x^{-1}, there exists a neighbourhood U of x such that $U^{-1} \subseteq W$.

It is easy to see that the mappings g_1 and g_2 are continuous in all their variables together if, and only if, the mapping

$g_3 : (x, y) \to xy^{-1}$

of $L \times L$ onto L is continuous.

Theorem 1. Let a be a fixed element of a semitopological lattice ordered group L. Then the mapping

$\gamma_a : x \to xa$

$l_a : x \to ax$

of L onto L are homeomorphisms of G.

Proof. It is clear that γ_a is a 1:1 and onto mapping. Let W be a neighbourhood of xa. Since L is a semitopological lattice ordered group, there exists a neighbourhood U of x such that $Ua \subseteq W$. This show that γ_a is continuous.

Moreover, it is easy to see that the inverse of γ_a^{-1} of γ_a is the mapping $x \to xa^{-1}$, which is continuous by the same argument as above. Hence γ_a is a homeomorphism. The fact that l_a is a homeomorphism follows similarly.

γ_a and l_a are respectively, called the right and left translations of L.

Corollary 1. Let F be closed, P an open, and A be any subset of a semitopological lattice ordered group L and let $a \in L$. Then

(i) aF and Fa are closed.

(ii) Pa, aP, AP and PA are open.

Proof. Since the mapping in theorem 1 are homeomorphisms, (i) is obvious. By the same argument, Pa and aP are open in (ii).

Since $AP = \bigcup_{a \in A} aP$,

$PA = \bigcup_{a \in A} Pa$

are the union of open sets and is therefore open. (ii) is established.

Corollary 2. Let L be a semitopological lattice ordered group. For any $x_1, x_2 \in L$, there exists a homeomorphisms f of G such that $f(x_1) = x_2$.

Proof. Let $x_1^{-1}x_2 = a \in L$ and consider the mapping $f : x \to xa$. Then f is a homeomorphism by theorem 1 and $f(x_1) = x_2$.

A lattice for which Corollary 2 is true is called homogeneous lattice space.

Reference

[1] G. Birkhoff : Lattice Theory, AMS Publication, reprinted 1984, p. 248.

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