ON THE CONSTRUCTION OFBALANCED INCOMPLETE BLOCK DESIGNS USING MOLS OF ORDER SIX – A SPECIAL CASE

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Abstract:The Balanced Incomplete Block Designs (BIBD)is play an important role in design of experiments and especially in the field of agriculturalexperiments.In this design ensure that treatments are compared with equal precision.The several methods are available in the literature to construction of BIBDs. In this paper, we have proposed a method for construction ofbalanced incomplete block designsusing Mutually Orthogonal LatinSquares (MOLS) of order six is a special case developed through Galois Field (GF) theory.

Key Words:*LSD, MOLS, Orthogonal, Galois Field, BIBD.*2014 Mathematics and Statistics Subject Classification: 62*k*10, 62*k*99.

1. INTRODUCTION

The balanced incomplete block design plays an important role in the field of agricultural experiments and it was first introduced by Yatesin 1936. In the designs, the allotments of k of the v treatments in different blocks, so that each pair of treatments is replicated a constant number of times. So, it hascome on constructional problems.The solutions of all known designs indicating impossible designs have been tabulated by Fisher and Yates (1938) produced tables of parameters for small BIBDs. There are several researchers proposed for different construction methods of BIBDs, some of them aregiven below.

R. C. Bose (1939) has developed to make a systematic study of the problem of construction of balanced incomplete block designs. It has been possible to obtain a number of new solutions, thus filling in some gaps in the tables of Fisher and Yates (1938). Also, it has been possible to exhibit the solutions for all the cases given there, together with the new solutions obtained, as members of certain general series of designs, the solutions to which can be obtained. The method of constructing such solutions is based on the concept of symmetrically repeated differences. By the use of this concept it is possible to construct the whole solution with the help of a few initial blocks. The discovery of the initial blocks is much facilitated by the use of properties of primitive roots of binomial equations in Galois fields, but the use of these properties is not essential. Whenever the initial blocks can be obtained by trial or otherwise, satisfying the condition that the differences are symmetrically repeated, a solution is obtainable of course, besides the series of designs discussed. There exist many other series of great theoretical interest. These have not been considered, as my object in this preliminary paper has been to obtain only a number of series, which should include between them all the known solvable cases with $r = 5$. Designs with $r > 10$ have been considered for illustrative purposes only. In all sets of values of v, *b, k, r, λ* with $r \leq 10$. Other than those corresponding to unreduced or orthogonal series solutions, or those for which a solution is definitely known not to exist, have been listed (the reference number used being the same as that used by Fisher and Yates (1938) in their tables).R.C. Bose (1954) was introduced in systematic methods for the construction of balanced incomplete block designs. It has been shown that these designs can be classified in a number of series, and that the general combinatorial solution for a design belonging to any series is obtainable, subject to certain conditions. In this way, besides obtaining by a systematic method the solutions for all the known solvable designs, it has been possible to obtain solutions for a number of new cases. Depending on the use of finite geometries, and the other the *Method of symmetrically repeated differences* explained and applied, supplemented by the processes of block section and block intersection.Kiefer and Wynn (1981) have discussed some methods for the construction of equineighbored balanced incomplete block designs. In this article an algorithm for constructing designs with $k = 3$ is developed. Ching-Shui Cheng(1983) has proposed the construction of optimal balanced incomplete block designs for correlated observations. Balanced incomplete block designs have been shown to be optimal for the elimination of one-way heterogeneity under homoscedastic and additive models.Das M.N andGiri (1986) have discussed about briefly explained the Latin Square Design, Graeco Latin Squares and Mutually Orthogonal Latin Square(MOLS) and the construction method of the above designs.Dharmalingam M (2002) has discussed the construction of partial Traillel crosses based on Trojan square design. In this paper the construction of Trojan square design of order $(9\times9)/3$ based on MOLS was carried out. Fariha Yasmin, Rashid Ahmed and Munir Akhtar (2015) have introduced the method of construction of balanced incomplete block designs using cyclic shifts.In this method, theyexpress some standard properties of a design just through examining the sets of shifts rather than studying the whole design.Jaisankar R and Pachamuthu M (2015) have discussed the general method of construction of BIBD and Partially Balanced Incomplete Block Designs (PBIBD)of some prime or prime power based onmutually orthogonal Latin square design. In this paper, we proposed a newmethodology has been given for the construction of balanced incomplete block designof order six is a special case using MOLS developed through*Galois Field* theory.

2. PRELIMINARIES

2.1 Definition of Latin Square Designs

A square array which contains *n* different elements with each element occurring *n* times but with no element occurring twice in the same column or row and which is used especially in the statistical design of experiments.An experiment designs that can be used to control the random variation of two factors. The design is arranged with an equal number of rows and columns, so that all combinations of possible values for the two variables can be tested multiple times. This design is used to reduce the effect of random or nuisance factors.In combinatory and in experimental design, a Latin Square is an $n \times n$ array filled with *n* different symbols, each occurring exactly once in each row and exactly once in each columnis known as *Latin Square Designs* (LSD).

2.2 Definition of BIBD

If in a block, the number of units or plots is smaller than the number of treatments, then the block is said to be incomplete $(b > k)$ and a design constituted of such blocks is called an incomplete block designs.An incomplete block design with *v* treatments distributed over *b* blocks, each of size *k*, where *k* is less than *v* such that each treatment occurs in *r* blocks, no treatment occurs more than once in a block and each pair of treatments occurs together in λ blocks, is called a *balanced incomplete block design*.

2.3 Parametric Relationships of BIBD

There are three parametric relationships of BIBD are given below;

1. $vr = bk = n$
2. $\lambda(v-1) = r(k-1)$
3. $b > v$.

2.4 Definition of MOLS

The concept of MOLS was first introduced by Euler in 1783 as Graeco - Latin or Euler squares. As stated earlier a Latin square of order *n* is an $(n \times n)$ array with entries $1, 2, \ldots, n$ having the property that each element of $1, 2, \ldots, n$ occurs exactly once in each row and column. Two Lain squares $A = (a_{ij})$ and $B = (b_{ij})$ of order *n* are said to orthogonal if, when they are super-imposed, for any $x, y \in 1, 2, \ldots, n$ there exists a unique position (i, j) such that $a_{ij} = x$ and $b_{ij} = y$. A set of Latin squares of same order are called MOLS, if every pair of Latin squares among them are orthogonal to each other.In the 1780s Eulerdemonstrated methods for constructing orthogonal Latin squares called Graeco - Latin squares where *n* is odd or a multiple of four. Observing that no order two square exists and unable to construct an order sixsquare, he conjectured that none exist for any oddly even number $n \equiv 2 \pmod 4$. Parker, Bose andShrikhande (1959) presented their paper showing Euler'sconjecture to be false for all $n \geq 10$. Thus, Orthogonal Latin squares exist for all orders $n \geq 3$ except $n = 6$. In this paper an attempt is made to construction of balanced incomplete block designs using mutually orthogonal Latin squares of orders six through Galois field theory.

3.THE CONSTRUCTION OF LSDs OF ORDER (6×6)

Let us consider the construction of LSDs of order six. The primitive elements of Galois Field elements *GF* (*p*) are; $0,1,2,\alpha,\alpha+1$ and $\alpha+2$. The principal row and column of the first Latin square is taken in natural order. Then the other elements are obtained by adding the corresponding entries of row and column. Then the table of elements of *GF* (2^2+2^1) can be formed as below.

3.1 First Summation LSD of the Elements of Order (6×6)

+	0	1	2	α	$\alpha+1$	$\alpha+2$
0	0	1	2	α	$\alpha+1$	$\alpha+2$
1	1	2	0	$\alpha+1$	$\alpha+2$	α
2	2	0	1	$\alpha+2$	α	$\alpha+1$
α	α	$\alpha+1$	$\alpha+2$	0	1	2
$\alpha+1$	$\alpha+1$	$\alpha+2$	α	1	2	0
$\alpha+2$	$\alpha+2$	α	$\alpha+1$	2	0	1

Substituting $2\alpha=\alpha$ and reduced to Mod 3 and 2α, the first Latin square is obtained as

3.2 First Latin Square of Order (6×6)

0	1	2	3	4	5
1	2	0	4	5	3
2	0	1	5	3	4
3	4	5	0	1	2
4	5	3	1	2	0
5	3	4	2	0	1

The principal column for the second Latin Square is obtained by multiplying the entries in the principal column of the first summation table by 2 and substituting $2\alpha=\alpha$. Then other elements are obtained by adding the corresponding entries of row and column.

3.3 Second Summation LSD of the Elements of Order (6×6)

+	0	1	2	α	$\alpha+1$	$\alpha+2$
0	0	1	2	α	$\alpha+1$	$\alpha+2$
2	2	0	1	$\alpha+2$	α	$\alpha+1$
1	1	2	0	$\alpha+1$	$\alpha+2$	α
α	α	$\alpha+1$	$\alpha+2$	0	1	2
$\alpha+2$	$\alpha+2$	α	$\alpha+1$	2	0	1
$\alpha+1$	$\alpha+1$	$\alpha+2$	α	1	2	0

Substituting $2\alpha=\alpha$ and reducing to mod 3 and 2α, the second Latin square is obtained as,

3.4 Second Latin Square of Order (6×6)

0	1	2	3	4	5
2	0	1	5	3	4
3	4	5	0	1	2
4	5	3	1	2	0
5	3	4	2	0	1
1	2	0	4	5	3

3.5 Third, Fourth and Fifth LSDs of Order (6×6)

Consider the first LSD of order six. We can place the 2nd and 3rd rows to the position of 5th and 6th rows respectively and the other rows moving upwards accordingly we get the third LSD of order six. Similarly, we can place the 4th row to the position of 6th row and the other rows moving upwards accordingly, we get the forth LSD of order six. Also, we can place the 5th row to the position of 6th row and the other rows moving upwards accordingly, we get the fifth LSD of order six. All these three Latin Squares (3rd, 4th and 5th LSDs) are given below;

0	1	2	3	4	5
3	4	5	0	1	2
4	5	3	1	2	0
5	3	4	2	0	1
1	2	0	4	5	3
2	0	1	5	3	4

0	1	2	3	4	5
4	5	3	1	2	0
5	3	4	2	0	1
1	2	0	4	5	3
2	0	1	5	3	4
3	4	5	0	1	2

0	1	2	3	4	5
5	3	4	2	0	1
1	2	0	4	5	3
2	0	1	5	3	4
3	4	5	0	1	2
4	5	3	1	2	0

3.6 The Construction of MOLS of Order (6×6)

Let the five LSDs of order six and superimposed these LSDs are given below

00000	11111	22222	33333	44444	55555
12345	20453	01534	45012	53120	34201
23451	04532	15340	50124	31205	42013
34512	45320	53401	01245	12053	20134
45123	53204	34015	12450	20531	01342
51234	32045	40153	24501	05312	13420

4. METHOD OF CONSTRUCTION OF BIBD

Assume that there are v treatments. Then using the Galois field primitive elements 0, 1,α, α^2, 2, 2α…,(v-1) MOLS can be obtained. Then by superimposing these (v-1) MOLS, one can get a Latin square like arrangement. One particular row/column of this arrangement would be consisting of a same sequence of treatment numbers in each cell. Then omit that row/column. Then from the resulting arrangement BIBDs can be constructed by two methods.

Method I

1. The BIBD can be obtained by treating each cell of the resulting arrangement as a block of BIBD.
2. The BIBD obtained by this method would be with block size (v-1). It can be noted that since one row / column of the super imposed LSD, there would be the $(v^2 - v)$ distinct blocks with block size (v-1) in the BIBD so obtained.
3. BIBDs with lesser block size can also be obtained in the similar way by superimposing two or more MOLS.
4. The number BIBDs with block size k would then be $(v-1)C_k$.

Method II

1. By separating each row of the resulting arrangement, one can see that each row constitutes a distinct BIBD and thus there would be (v-1) BIBDs with block size(v-1).

4.1 Construction of BIBD by method I using MOLS of order six

Consider the MOLS of order six and by superimposing these MOLS the following structure is obtained.(Section 3.2) are given below

~~00000~~	~~11111~~	~~22222~~	~~33333~~	~~44444~~	~~55555~~
12345	20453	01534	45012	53120	34201
23451	04532	15340	50124	31205	42013
34512	45320	53401	01245	12053	20134
45123	53204	34015	12450	20531	01342
51234	32045	40153	24501	05312	13420

Since the first row consists of the cell with same treatments it is to be omitted. BIBD can be obtained from the remaining five rows as below.

Blocks	***Treatments***					***Blocks***	***Treatments***					***Blocks***	***Treatments***				
1	1	2	3	4	5	**11**	3	1	2	0	5	**21**	3	4	0	1	5
2	2	0	4	5	3	**12**	4	2	0	1	3	**22**	1	2	4	5	0
3	0	1	5	3	4	**13**	3	4	5	1	2	**23**	2	0	5	3	1
4	4	5	0	1	2	**14**	4	5	3	2	0	**24**	0	1	3	4	2
5	5	3	1	2	0	**15**	5	3	4	0	1	**25**	5	1	2	3	4
6	3	4	2	0	1	**16**	0	1	2	4	5	**26**	3	2	0	4	5
7	2	3	4	5	1	**17**	1	2	0	5	3	**27**	4	0	1	5	3
8	0	4	5	3	2	**18**	2	0	1	3	4	**28**	2	4	5	0	1
9	1	5	3	4	0	**19**	4	5	1	2	3	**29**	0	5	3	1	2
10	5	0	1	2	4	**20**	5	3	2	0	4	**30**	1	3	4	2	0

In the above table,v = Number of treatments = 6, b = Number ofblocks = 30, r = Number of replicates = 25, k = Size of the block = 5 and n = Total number of blocks = 150. Here $k<v$(5 < 6), above problem is incomplete design. The parametric relationships are satisfied of BIBD are given below;

1. $vr = bk = n$

$6\times25 = 30\times5 = 150$

$150 = 150 = 150.$

2. λ (v-1) = r(k-1), λ (5) = 100, λ = 20. Here, all possible pairs of treatments are occurringexactly λ = 6 times in blocks.

20×5 = 25×4, 100 = 100.

3. $b>v$ (Fishers Inequality)

   30 > 6.

4.2Construction of BIBD by using Method II separating rows of order six

~~00000~~	~~11111~~	~~22222~~	~~33333~~	~~44444~~	~~55555~~
12345	20453	01534	45012	53120	34201
23451	04532	15340	50124	31205	42013
34512	45320	53401	01245	12053	20134
45123	53204	34015	12450	20531	01342
51234	32045	40153	24501	05312	13420

Since the first row consists of the cell with same treatments it is to be omitted. The five different BIBDs can be obtained and given below;

BIBD 1						BIBD 2						BIBD 3					
Blocks	**Treatments**					**Blocks**	**Treatments**					**Blocks**	**Treatments**				
1	1	2	3	4	5	**1**	2	3	4	5	1	**1**	3	4	5	1	2
2	2	0	4	5	3	**2**	0	4	5	3	2	**2**	4	5	3	2	0
3	0	1	5	3	4	**3**	1	5	3	4	0	**3**	5	3	4	0	1
4	4	5	0	1	2	**4**	5	0	1	2	4	**4**	0	1	2	4	5
5	5	3	1	2	0	**5**	3	1	2	0	5	**5**	1	2	0	5	3
6	3	4	2	0	1	**6**	4	2	0	1	3	**6**	2	0	1	3	4

BIBD 4						BIBD 5					
Blocks	**Treatments**					**Blocks**	**Treatments**				
1	4	5	1	2	3	**1**	5	1	2	3	4
2	5	3	2	0	4	**2**	3	2	0	4	5
3	3	4	0	1	5	**3**	4	0	1	5	3
4	1	2	4	5	0	**4**	2	4	5	0	1
5	2	0	5	3	1	**5**	0	5	3	1	2
6	0	1	3	4	2	**6**	1	3	4	2	0

In the above table, each BIBDs has$v$ = Number of treatments = 6, b = Number of blocks = 6, r = Number of replicates = 5, k = Size of the block = 5. n = Total number of blocks = 30. Here$k<v$(5 < 6), above problem is incomplete design. The parametric relationships satisfied the parametric relations of BIBD are given below;

1. $vr = bk = n$

   $6\times5 = 6\times5 = 30$

   $30 = 30 = 30.$

2. $\lambda (v\text{-}1) = r (k\text{-}1)$, $\lambda (5) = 20$, $\lambda = 4$. Here, all possible pairs of treatments are occurring exactly $\lambda = 4$ times in blocks.

   $4\times5 = 5\times4$, $20 = 20$.

3. $b>v$ (Fishers Inequality)

   $6> 6$.

5. CONCLUSION

The method of construction is discussed in this paper is based on MOLS developed through Galois Field theory. Based on this theory it is proved that a balanced incomplete blockdesigns can be constructed sixth order a special case, the number of treatments is a non-prime. Similar methodology can be followed for the construction of MOLS of any order which is a non-prime number.

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