OBSERVATION ON PELL EQUATION $y^2=14x^2+4$

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ABSTRACT: The binary quadratic equation $y^2=14x^2+4$ is considered and a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration a few patterns of Pythagorean triangles and rectangles are observed.

KEYWORDS: Binary Quadratic, Integral Solutions, Polygonal numbers, Pyramidal numbers.

NOTATIONS:

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

Pr_n : Pronic number of rank n

S_n : Star number of rank n

$Ct_{m,n}$: Centered pyramidal number of rank n with size m

I. Introduction

The Binary Quadratic equation of the form $y^2=Dx^2+1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1,2,3,4]. In [5] infinitely many Pythagorean triangles in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation $Y^2=3X^2+1$. In [6], a special Pythagorean triangle is obtained by employing the integral solutions of $Y^2=10X^2+1$. In [7] different patterns of infinitely many Pythagorean triangles are obtained by employing the non-integral solutions of $Y^2=5X^2+1$. In this context one may also refer [8-11]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2=14x^2+4$ representing a hyperbola. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The positive pell equation representing hyperbola under consideration is

$$y^2=14x^2+4 \qquad (1.1)$$

whose smallest positive integer solution is

$$x_0=8, y_0=30$$

To obtain the other solutions of (1.1), consider the pell equation

$$y^2=14x^2+1$$

with initial solution is

$$\tilde{x}_0=4, \tilde{y}_0=15$$

whose general solutions is given by

$$\tilde{x}_n=\frac{1}{2\sqrt{14}}g_n, \tilde{y}_n=\frac{1}{2}f_n$$

where

$$f_n = (15+4\sqrt{14})^{n+1} + (15-4\sqrt{14})^{n+1}$$
$$g_n = (15+4\sqrt{14})^{n+1} - (15-4\sqrt{14})^{n+1}$$

Applying brahmagupta lemma between $(\tilde{x_0}, \tilde{y_0})$ and $(\tilde{x_n}, \tilde{y_n})$, the other integer solutions of(1.1) are given by

$$x_{n+1} = 4f_n + \frac{15}{\sqrt{14}} g_n \quad (1.2)$$

$$y_{n+1} = 15f_n + 4\sqrt{14}\, g_n \quad (1.3)$$

TO FIND THE RECURRENCE RELATION OF y_{n+1}:

Replacing n+1 by n+2 and n+3 in (1.3) in turn, we get

$$y_{n+2} = 449f_n + 120\sqrt{14}\, g_n \quad (1.4)$$

$$y_{n+3} = 13455f_n + 3596\sqrt{14}\, g_n \quad (1.5)$$

From (1.3),(1.5)and(1.6) we get,

$$y_{n+1} - 30y_{n+2} + y_{n+3} = 0$$

TO FIND THE RECURRENCE RELATION OF x_{n+1} :

Replacing n+1 by n+2 and n+3 in (1.2) in turn, we get

$$x_{n+2} = 120f_n + \frac{449}{\sqrt{14}} g_n \quad (1.6)$$

$$x_{n+3} = 3596f_n + \frac{13455}{\sqrt{14}} g_n \quad (1.7)$$

From (1.2),(1.6)and(1.7) we get,

$$x_{n+1} - 30x_{n+2} + x_{n+3} = 0$$

Some numerical examples are given in the following table below:

n	x_n	y_n
0	8	30
1	240	898
2	7192	26910
3	215520	806402
4	6458408	24165150
5	193536720	724148098
6	5799643192	21700277790
7	173795759040	650284185602
8	5208073128008	19486825290270
9	156068398081200	583954474522498
10	4676843869307992	17499147410384670

From the above table, we observe some interesting relations among the solutions which are presented below:

A few interesting relations are given below:

- $x_{n+3} = 30x_{n+2} - x_{n+1}$
- $4y_{n+1} = x_{n+2} - 15x_{n+1}$
- $y_{n+2} = 15x_{n+2} - x_{n+1}$
- $4y_{n+3} = 449x_{n+2} - 15x_{n+1}$
- $120y_{n+1} = x_{n+3} - 449x_{n+1}$
- $8y_{n+2} = x_{n+3} - x_{n+1}$
- $120y_{n+3} = 449x_{n+3} - x_{n+1}$
- $4y_{n+1} = 15x_{n+3} - 449x_{n+2}$
- $4y_{n+2} = x_{n+3} - 15x_{n+2}$
- $4y_{n+3} = 15x_{n+3} - x_{n+2}$
- $56x_{n+1} = y_{n+2} - 15y_{n+1}$
- $56x_{n+2} = 15y_{n+2} - y_{n+1}$
- $56x_{n+3} = 449y_{n+2} - 15x_{n+1}$
- $y_{n+3} = 30y_{n+2} - y_{n+1}$
- $1680x_{n+1} = y_{n+3} - 449y_{n+1}$
- $112x_{n+2} = y_{n+3} - y_{n+1}$
- $1680x_{n+3} = 449y_{n+3} - y_{n+1}$
- $1680y_{n+2} = 56y_{n+1} + 56y_{n+3}$
- $56x_{n+1} = 15y_{n+3} - 449y_{n+2}$
- $56x_{n+2} = y_{n+3} - 15y_{n+2}$
- $56x_{n+3} = 15y_{n+3} - y_{n+2}$
- $15y_{n+1} = y_{n+2} - 56x_{n+1}$
- $449x_{n+2} = 4y_{n+3} + 15x_{n+1}$
- $15x_{n+3} = 4y_{n+1} + 449y_{n+2}$
- $15x_{n+3} = 4y_{n+3} + x_{n+2}$
- $15y_{n+1} = 15y_{n+3} - 1680x_{n+2}$
- $15x_{n+1} = 15x_{n+3} - 120y_{n+2}$
- $15y_{n+1} = 499y_{n+2} - 56x_{n+3}$

Each of the following expression is a nasty number

- $6(15y_{2n+2} - 56x_{2n+2} + 2)$
- $6(y_{2n+3} - 112x_{2n+2} + 2)$
- $2694(15y_{2n+4} - 50344x_{2n+2} + 898)$
- $10(449y_{2n+2} - 56x_{2n+3} + 30)$
- $6(449y_{2n+3} - 1680x_{2n+3} + 2)$
- $10(449y_{2n+4} - 50344x_{2n+3} + 30)$
- $2694(13455y_{2n+2} - 56x_{2n+4} + 898)$

- $10(13455y_{2n+3} - 1680x_{2n+4} + 30)$
- $6(13455y_{2n+4} - 50344x_{2n+4} + 2)$
- $6(15x_{2n+3} - 449x_{2n+2} + 8)$
- $3(x_{2n+4} - 897x_{2n+2} + 16)$
- $6(449x_{2n+4} - 13455x_{2n+3} + 8)$
- $6(30y_{2n+2} - y_{2n+3} + 2)$
- $5(899y_{2n+2} - y_{2n+4} + 60)$
- $21(50344y_{2n+3} - 1680y_{2n+4} + 112)$

Each of the following expression is a cubic integer

- $15y_{3n+3} - 56x_{3n+3} + 3(y_{n+2} - 112x_{n+1})$
- $y_{3n+4} - 112x_{3n+3} + 3(15y_{n+1} - 56x_{n+1})$
- $(449)^2(15y_{3n+5} - 50344x_{3n+3} + 1347(449y_{n+2} - 1680x_{n+2}))$
- $(15)^2(449y_{3n+3} - 56x_{3n+4} + 45(15y_{n+1} - 56x_{n+1}))$
- $449y_{3n+4} - 1680x_{3n+4} + 3(13455y_{n+3} - 50344x_{n+3})$
- $(15)^2(449y_{3n+5} - 50344x_{3n+4} + 45(15y_{n+1} - 56x_{n+1}))$
- $(449)^2(13455y_{3n+3} - 56x_{3n+5} + 1347(13455y_{n+1} - 56x_{n+3}))$
- $(15)^2(13455y_{3n+4} - 1680x_{3n+5} + 45(449y_{n+3} - 50344x_{n+2}))$
- $13455y_{3n+5} - 50344x_{3n+5} + 3(15y_{n+1} - 56x_{n+1})$
- $(4)^2(15x_{3n+4} - 449x_{3n+3} + 3(449x_{n+3} - 13455x_{n+2}))$
- $x_{3n+5} - 897x_{3n+3} + 3(x_{n+3} - 897x_{n+1})$
- $2(449x_{3n+5} - 13455x_{3n+4} + 3(15x_{n+2} - 449x_{n+1}))$
- $30y_{3n+3} - y_{3n+4} + 3(15y_{n+1} - 56x_{n+1})$
- $(30)^2(899y_{3n+3} - y_{3n+5} + 6(449y_{n+1} - 56x_{n+2}))$
- $(56)^2(50344y_{3n+4} - 1680y_{3n+5} + 168(y_{n+2} - 112x_{n+1}))$

Each of the following expression is a biquadratic integer

- $15y_{4n+4} - 56x_{4n+4} + 4(y_{n+2} - 112x_{n+1})^2 - 2$
- $(449)^3(15y_{4n+6} - 50344x_{4n+4} + 1796(15y_{n+1} - 56x_{n+1})^2)$
- $y_{4n+5} - 112x_{4n+4} + 4(449y_{n+2} - 1680x_{n+2})^2 - 2$
- $(15)^3(449y_{4n+4} - 56x_{4n+5} + 60(449y_{n+2} - 1680x_{n+2})^2 - 30)$
- $449y_{4n+5} - 1680x_{4n+5} + 4(13455y_{n+3} - 50344x_{n+3})^2 - 2$
- $(15)^3(449y_{4n+6} - 50344x_{4n+5} + 60(15y_{n+1} - 56x_{n+1})^2 - 30)$
- $(449)^3(13455y_{4n+4} - 56x_{4n+6} + 1796(y_{n+2} - 112x_{n+1})^2 - 898)$
- $(15)^3(13455y_{4n+5} - 1680x_{4n+6} + 60(449y_{n+2} - 1680x_{n+2})^2 - 30)$

- $13455y_{4n+6} - 50344x_{4n+6} + 4(15y_{n+1} - 56x_{n+1})^2 - 2$
- $(2)^2(15x_{4n+5} - 449x_{4n+4} + (449x_{n+3} - 13455x_{n+2})^2 - 8)$
- $2(x_{4n+6} - 897x_{4n+4} + 2(15x_{n+2} - 449x_{n+1})^2 - 16)$
- $(2)^2(449x_{4n+6} - 13455x_{4n+5} + 16(15y_{n+1} - 56x_{n+1})^2 - 8)$
- $30y_{4n+4} - y_{4n+5} + 4(13455y_{n+3} - 50344x_{n+3})^2 - 2$
- $(30)^3(899y_{4n+4} - y_{4n+6} + 120(30y_{n+1} - y_{n+2})^2 - 60)$
- $(56)^3(50344y_{4n+5} - 1680y_{4n+6} + 224(30y_{n+1} - y_{n+2})^2 - 112)$

Each of the following expression is a quintic integer

- $15y_{5n+5} - 56x_{5n+5} + 5(y_{n+2} - 112x_{n+1})^3 - 5(449y_{n+2} - 1680x_{n+2})$
- $y_{5n+6} - 112x_{5n+5} + 5(15y_{n+1} - 56x_{n+1})^3 - 5(y_{n+2} - 112x_{n+1})$
- $(449)^4(15y_{5n+7} - 50344x_{5n+5} + 2245(449y_{n+2} - 1680x_{n+2})^3 - 2245(15y_{n+1} - 56x_{n+1}))$
- $(15)^4(449y_{5n+5} - 56x_{5n+6} + 75(449y_{n+2} - 1680x_{n+2})^3 - 75(15y_{n+1} - 56x_{n+1}))$
- $449y_{5n+6} - 1680x_{5n+6} + 5(13455y_{n+3} - 50344x_{n+3})^3 - 5(449y_{n+2} - 1680x_{n+2})$
- $(15)^4(449y_{5n+7} - 56x_{5n+6} + 75(449y_{n+2} - 1680x_{n+2})^3 - 75(15y_{n+1} - 56x_{n+1}))$
- $(449)^4(13455y_{5n+5} - 56x_{5n+7} + 2245(15y_{n+1} - 56x_{n+1})^3 - 2245(13455y_{n+3} - 50344x_{n+3}))$
- $(15)^4(13455y_{5n+6} - 1680x_{5n+7} + 75(13455y_{n+3} - 50344x_{n+3})^3 - 75(y^{n+2} - 112x_{n+1}))$
- $13455y_{5n+7} - 50344x_{5n+7} + 5(15y_{n+1} - 56x_{n+1})^3 - 5(y_{n+2} - 112x_{n+1})$
- $(4)^4\ (15x_{5n+6} - 449x_{5n+5} + 20(30y_{n+1} - y_{n+2})^3 - 20(449y_{n+2} - 1680x_{n+2}))$
- $(2)^2(x_{5n+7} - 897x_{5n+5} + 40(y_{n+2} - 112x_{n+1})^3 - 10(15x_{n+2} - 449x_{n+1}))$
- $(2)^3\ (449x_{5n+7} - 13455x_{5n+6} + 20(449y_{n+2} - 1680x_{n+2})^3 - 5(15x_{n+2} - 449x_{n+1}))$
- $30y_{5n+5} - y_{5n+6} + 5(15x_{n+2} - 449x_{n+1})^3 - 5(30y_{n+1} - y_{n+2})$
- $(30)^4\ (899y_{5n+5} - y_{5n+7} + 150(y_{n+2} - 112x_{n+1})^3 - 10(449y_{n+1} - 56x_{n+2}))$
- $(56)^4\ (50344y_{5n+6} - 1680y_{5n+7} + 280(30y_{n+1} - y_{n+2})^3 - 70(15x_{n+2} - 449x_{n+1}))$

III. Remarkable Observations

1. Employing the linear combinations among the solutions of (1.1), one may generate integer solution for other choices of hyperbola which are presented in the table below.

S.No	Hyperbola	(X_n, Y_n)
1.	$X_n^2 - 14Y_n^2 = 4$	$X_n = 15y_{n+1} - 56x_{n+1}$ $Y_n = 15x_{n+1} - 4y_{n+1}$

2.	$225X_n^2 - 14Y_n^2 = 900$	$X_n = y_{n+2} - 112x_{n+1}$ $Y_n = 449x_{n+1} - 4y_{n+2}$
3.	$X_n^2 - 14Y_n^2 = 806404$	$X_n = y_{n+2} - 112x_{n+1}$ $Y_n = 449x_{n+1} - 4y_{n+2}$
4.	$X_n^2 - 3150Y_n^2 = 900$	$X_n = 449y_{n+1} - 56x_{n+2}$ $Y_n = x_{n+2} - 8y_{n+1}$
5.	$X_n^2 - 14Y_n^2 = 4$	$X_n = 449y_{n+2} - 1680x_{n+2}$ $Y_n = 449x_{n+2} - 120y_{n+2}$
6.	$X_n^2 - 14Y_n^2 = 900$	$X_n = 449y_{n+3} - 50344x_{n+2}$ $Y_n = 13455x_{n+2} - 120y_{n+3}$
7.	$X_n^2 - 14Y_n^2 = 806404$	$X_n = 13455y_{n+1} - 56x_{n+3}$ $Y_n = 15x_{n+3} - 3596y_{n+1}$
8.	$X_n^2 - 14Y_n^2 = 900$	$X_n = 13455y_{n+2} - 1680x_{n+3}$ $Y_n = 449x_{n+3} - 3596y_{n+2}$
9.	$X_n^2 - 14Y_n^2 = 4$	$X_n = 13455y_{n+3} - 50344x_{n+3}$ $Y_n = 13455x_{n+3} - 3596y_{n+3}$
10.	$X_n^2 - 224Y_n^2 = 64$	$X_n = 15x_{n+2} - 449x_{n+1}$ $Y_n = 30x_{n+1} - x_{n+2}$
11.	$900X_n^2 - 896Y_n^2 = 230400$	$X_n = x_{n+3} - 897x_{n+1}$ $Y_n = 899x_{n+1} - x_{n+3}$
12.	$X_n^2 - 14Y_n^2 = 64$	$X_n = 499x_{n+3} - 13455x_{n+2}$ $Y_n = 3596x_{n+2} - 120x_{n+3}$
13.	$3136X_n^2 - 14Y_n^2 = 12544$	$X_n = 30y_{n+1} - y_{n+2}$ $Y_n = 15y_{n+2} - 449y_{n+1}$
14.	$3136X_n^2 - 3150Y_n^2 = 11289600$	$X_n = 899y_{n+1} - y_{n+3}$ $Y_n = y_{n+3} - 897y_{n+1}$

15.	$X_n^2 - 14Y_n^2 = 12544$	$X_n = 50344y_{n+2} - 1680y_{n+3}$ $Y_n = 449y_{n+3} - 13455y_{n+2}$

2. Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of parabola which are presented in the table below:

S.No	Parabola	(X_n, Y_n)
1.	$X_n - 14Y_n^2 = 4$	$X_n = 15y_{2n+2} - 56x_{2n+2} + 2$ $Y_n = 15x_{n+1} - 4y_{n+1}$
2.	$225X_n - 14Y_n^2 = 900$	$X_n = y_{2n+3} - 112x_{2n+2} + 2$ $Y_n = 449x_{n+1} - 4y_{n+2}$
3.	$449X_n - 14Y_n^2 = 806404$	$X_n = 15y_{2n+4} - 50344x_{2n+2} + 898$ $Y_n = x_{n+2} - 8y_{n+1}$
4.	$X_n - 210Y_n^2 = 60$	$X_n = 449y_{2n+2} - 56x_{2n+3} + 30$ $Y_n = x_{n+2} - 8y_{n+1}$
5.	$X_n - 14Y_n^2 = 4$	$X_n = 449y_{2n+3} - 1680x_{2n+3} + 2$ $Y_n = 449x_{n+2} - 120y_{n+2}$
6.	$15X_n - 14Y_n^2 = 900$	$X_n = 449y_{2n+4} - 50344x_{2n+3} + 30$ $Y_n = 13455x_{n+2} - 120y_{n+3}$
7.	$499X_n - 14Y_n^2 = 806404$	$X_n = 13455y_{2n+2} - 56x_{2n+4} + 898$ $Y_n = 15x_{n+3} - 3596y_{n+1}$
8.	$15X_n - 14Y_n^2 = 900$	$X_n = 13455y_{2n+3} - 1680x_{2n+4} + 30$ $Y_n = 449x_{n+3} - 3596y_{n+2}$
9.	$X_n - 14Y_n^2 = 4$	$X_n = 13455y_{2n+4} - 50344x_{2n+4} + 2$ $Y_n = 13455x_{n+3} - 3596y_{n+3}$

10.	$X_n - 56Y_n^2 = 16$	$X_n = 15x_{2n+3} - 449x_{2n+4} + 8$ $Y_n = 30x_{n+1} - x_{n+3}$
11.	$900X_n - 112Y_n^2 = 7200$	$X_n = x_{2n+4} - 897x_{2n+2} + 16$ $Y_n = 899x_{n+1} - x_{n+3}$
12.	$4X_n - 14Y_n^2 = 64$	$X_n = 499x_{2n+4} - 13455x_{2n+3} + 8$ $Y_n = 3596x_{n+2} - 120x_{n+3}$
13.	$X_n - 14Y_n^2 = 12544$	$X_n = 30y_{2n+2} - y_{2n+3} + 2$ $Y_n = 15y_{n+2} - 449y_{n+1}$
14.	$12544X_n - 420Y_n^2 = 1505280$	$X_n = 899y_{2n+2} - y_{2n+4} + 60$ $Y_n = y_{n+3} - 897y_{n+1}$
15.	$56X_n - 14Y_n^2 = 12544$	$X_n = 50344y_{2n+3} - 1680y_{2n+4} + 112$ $Y_n = 449y_{n+3} - 13455y_{n+2}$

3. Employing linear combinations among the solutions of (1.1),one may generate integer solutions for other choices of parabola which are presented in the table below:

S.No	Straight line	(X, Y)
1.	$Y = 449X$	$X = 15y_{n+1} - 56x_{n+1}$ $Y = 15y_{n+3} - 50344x_{n+1}$
2.	$Y = 449X$	$X = y_{n+2} - 112x_{n+1}$ $Y = 15y_{n+3} - 50344x_{n+1}$
3.	$Y = X$	$X = y_{n+2} - 112x_{n+1}$ $Y = 15y_{n+1} - 56x_{n+1}$

4.	$Y = 15X$	$X = y_{n+2} - 112x_{n+1}$ $Y = 449y_{n+1} - 56x_{n+2}$
5.	$Y = 15X$	$X = 449y_{n+2} - 1680x_{n+2}X$ $Y = 449y_{n+1} - 56x_{n+2}$
6.	$Y = 15X$	$X = y_{n+2} - 112x_{n+1}$ $Y = 449y_{n+1} - 56x_{n+2}$
7.	$Y = X$	$X = 449y_{n+1} - 56x_{n+2}$ $Y = 449y_{n+3} - 50344x_{n+2}$
8.	$Y = 15X$	$X = 449y_{n+2} - 1680x_{n+2}$ $Y = 449y_{n+3} - 50344x_{n+2}$
9.	$Y = 15X$	$X = y_{n+2} - 50344x_{n+2}$ $Y = 449y_{n+3} - 50344x_{n+2}$
10.	$Y = 15X$	$X = 15y_{n+1} - 56x_{n+1}$ $Y = 449y_{n+3} - 50344x_{n+2}$
11.	$Y = 449X$	$X = 449y_{n+2} - 1680x_{n+2}$ $Y = 13455y_{n+1} - 56x_{n+3}$
12.	$Y = 15X$	$X = 449y_{n+2} - 1680x_{n+2}$ $Y = 13455y_{n+2} - 1680x_{n+3}$
13.	$Y = 449X$	$X = 15y_{n+1} - 56x_{n+1}$ $Y = 13455y_{n+1} - 56x_{n+3}$
14.	$Y = X$	$X = 449y_{n+3} - 50344x_{n+2}$ $Y = 13455y_{n+2} - 1680x_{n+3}$

4. Consider $p = x_{n+1} + y_{n+1}$, $q = x_{n+1}$ observe that $p > q > 0$, Treat p,q as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$, where, $\alpha = 2pq$, $\beta = p^2 - q^2$, $\gamma = p^2 + q^2$

Then the following interesting relations are observed

i. $\alpha - 7\beta + 6\gamma = 4$

ii. $\frac{2A}{p} = x_{n+1} y_{n+1}$

iii. $\frac{Z-Y}{16}$ is a cubical number

iv. $\frac{Z-Y}{8}$ is a biquadratic number

v. $X - \frac{4A}{p} + Y$ is written as the sum of two squares

where A and P are represent area and perimeter of triangle T(α, β, γ)

5. Employing the solutions of (1.1),each of the following among the special polygonal, pyramidal ,star numbers,pronic numbers is a perfect square

- ❖ $\left(\frac{3p_{y-2}^{3}}{t_{3,y-2}}\right)^{2} - 14\left(\frac{6p_{x}^{3}}{pr_{x+1}}\right)^{2}$
- ❖ $\left(\frac{12p_{y}^{5}}{s_{y+1}-1}\right)^{2} - 14\left(\frac{36p_{x-2}^{3}}{s_{x-1}-1}\right)^{2}$
- ❖ $\left(\frac{p_{y}^{5}}{t_{3,y}}\right)^{2} - 14\left(\frac{12p_{x}^{5}}{s_{x+1}-1}\right)^{2}$
- ❖ $\left(\frac{6p_{y-1}^{4}}{t_{3,2y-1}}\right)^{2} - 14\left(\frac{4p_{x}^{5}}{ct_{4,x}-1}\right)^{2}$
- ❖ $\left(\frac{3p_{y-2}^{3}}{t_{3,y-2}}\right)^{2} - 14\left(\frac{p_{x}^{5}}{t_{3,x}}\right)^{2}$
- ❖ $\left(\frac{4p_{y}^{5}}{ct_{4,y}-1}\right)^{2} - 14\left(\frac{6p_{x}^{5}}{ct_{6,x}-1}\right)^{2}$
- ❖ $\left(\frac{6p_{y-1}^{4}}{t_{3,2y-1}}\right)^{2} - 14\left(\frac{3(p_{x+1}^{4} - p_{x+1}^{3}}{t_{4,x+1}}\right)^{2}$
- ❖ $\left(\frac{3p_{y-2}^{3}}{t_{3,y-2}}\right)^{2} - 14\left(\frac{4p_{x}^{5}}{ct_{4,x}-1}\right)^{2}$
- ❖ $\left(\frac{6p_{y-1}^{4}}{t_{3,2y-1}}\right)^{2} - 14\left(\frac{4pt_{x-3}}{p_{x-3}^{3}}\right)^{2}$

- $\left(\frac{2p_{y-1}^5}{t_{4,y-1}}\right)^2 - 14\left(\frac{3p_x^3}{t_{3,x+1}}\right)^2$

IV. CONCLUSION:

In this dissertation, we have presented infinitely many integer solutions for the hyperbola presented by the positive Pell equation and as the binary quadratic Diophantine equation are rich in variety, one may search for the other choices of positive Pell equations and determine their integral solution along with suitable properties.

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