Power Dominator Coloring of Certain Classes of Graphs

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Abstract : Let $\boldsymbol{G(V,E)}$ be a graph. A proper coloring of a graph $\boldsymbol{G}$ assigns colors to the vertices of $\boldsymbol{G}$, such that any two vertices joined by an edge are given different colors. A dominator coloring of a graph $\boldsymbol{G}$ is a proper coloring such that every vertex dominates all the vertices in at least one color class. Based on the concepts of dominator coloring and power domination, a new coloring concept called power dominator coloring of a graph G was introduced. The power dominator chromatic number $\boldsymbol{\chi_{pd}(G)}$ is the minimum number of colors required for a power dominator coloring of a graph $\boldsymbol{G}$. In this paper, we obtain the power dominator chromatic number of certain classes of graphs.

***Index Terms* – Graph Coloring, power domination, power dominator coloring, power dominator chromatic number.**

I. INTRODUCTION

A graph $G = (V, E)$ is a discrete structure with a finite set of objects, called vertices and a finite set of pairs of vertices, called edges. A proper coloring [2] of a graph G assigns colors to the vertices of G in such a way that any two adjacent vertices of G are assigned different colors. The chromatic number $\chi(G)$ is the minimum number of colors needed for a proper coloring of G. Based on the well-investigated notion of domination [4, 10] in graphs, the concept of dominator coloring [1, 3, 6, 7] of a graph G was introduced. A dominator coloring of a graph G is a proper coloring of the vertices of G, satisfying the property that every vertex dominates at least one color class which consists of those vertices of G that have the same color. The concept of power domination in a graph was introduced by Haynes et al. [9, 10] in modelling by graphs, the task of monitoring the state of an electric power system. The notion of power domination of vertices by a given vertex u can be described by associating a monitoring set $M(u)$ as follows: For a vertex u in a graph G, *(i)* $M(u) = N[u]$, the closed neighbourhood of u in G *(ii)* consider a vertex w which is originally not in $M(u)$ and add this vertex w to $M(u)$, whenever w has a neighbour $v \in M(u)$ such that all the neighbours of v other than w, are already in $M(u)$ *(iii)* repeat Step*(ii)* until no more vertex could be added to $M(u)$. Then we say that u power dominates the vertices in $M(u)$. Combining the concepts of dominator coloring and power domination, a new variant of proper coloring called power dominator coloring of a graph G was introduced [11,12] which requires that every vertex of the vertex set V power dominates all vertices of at least one color class. The power dominator chromatic number $\chi_{pd}(G)$ is the minimum number of colors required for a power dominator coloring of G. Certain properties of $\chi_{pd}(G)$ were derived in [11, 12], besides computing this number for certain classes of graphs. Here we compute $\chi_{pd}(G)$ for certain other special classes of graphs. For standard notions related to graphs and for special classes of graphs, we refer to [2, 5, 8, 11, 13, 14].

II. THE POWER DOMINATOR CHROMATIC NUMBER OF SPECIAL GRAPH CLASSES

We first consider three classes of graphs.

Definition 2.1

(*i*) A Tadpole graph $T(m,n)$ is the graph [14] obtained by joining by an edge, a vertex of the cycle C_m, $m \geq 3$, and an end vertex of the path P_n, $n \geq 1$.
(*ii*) A spider [5] is a rooted tree in which each vertex has degree one or two, except for the root which is of degree at least 3. A leg of a spider is a path from the root to a vertex of degree one.
(*iii*) The *m*-book graph B_m is the graph [13] $S_{m+1} \times P_2$, the Cartesian product of S_{m+1} and P_2 where S_{m+1} is the star graph and P_2 is the path on two vertices.

Theorem 2.1

(i) For the Tadpole graph $T_{m,n}$ $(m \geq 3, n \geq 1)$, the power dominator chromatic number is 3 *i.e.* $\chi_{pd}(T_{m,n}) = 3$
(ii) For a Spider graph S_n of order $n \geq 5$ and having at least one leg of length 2, the power dominator chromatic number is 3 *i.e.* $\chi_{pd}(S_n) = 3$
(iii) For the *m*- book graph B_m, with $m \geq 3$, the power dominator chromatic number is 3 *i.e.* $\chi_{pd}(B_m) = 3$.

Proof:

(i) Let the vertex set of the Tadpole graph $T_{m,n}$ $(m \geq 3, n \geq 1)$ be $V(T_{m,n}) = \{v_1, v_2, \ldots, v_m\} \cup \{u_1, u_2, \ldots, u_n\}$ where v_i $(1 \leq i \leq m)$ is on the cycle of the tadpole and u_i $(1 \leq i \leq n)$ is on the path. Let v_m be the vertex joined to the end vertex u_1 of the path in the Tadpole graph. Assign color 1 to the vertex v_m. The remaining vertices $v_i, 1 \leq i \leq m-1$ and $u_j, 1 \leq j \leq n$ are colored by new colors 2 for odd i, j and 3 for even i, j. Each vertex of $T_{m,n}$ power dominates the vertex v_m with color 1 which is the only vertex in the color class 1 . Note that the vertices v_1, v_{m-1}, u_1 being adjacent to v_m dominate and hence power dominate the vertex v_m . Each of the remaining vertices (on the cycle and on the path) power dominates v_m. For instance, the vertex v_2 on the cycle is adjacent to v_1 and hence dominates v_1 which in turn dominates the only other vertex v_m and so, by the definition of power domination, v_2 power dominates v_m. Thus $\chi_{pd}(T_{m,n}) = 3$.

(*ii*) We consider a spider graph of order $n \geq 5$ and having at least one leg of length 2. Now the root vertex is given color 1 and in a leg of the spider, which is a path, the vertices are given colors 2, 3 alternatively. It is clear that the vertices with colors 2 and 3 power dominate the root vertex and hence the color class 1. Also the root vertex with color 1 power dominates its own color. Hence the power dominator chromatic number of a spider graph is 3.

(iii) The m-book graph B_m is the Cartesian product $S_{m+1} \times P_2$ where S_{m+1} is the star graph with a central vertex v_1 and the m pendant vertices v_i , $2 \leq i \leq m+1$ and P_2 is the path on two vertices u_1, u_2 . The vertex set $V(B_m)$ of the m-book graph B_m consists of vertices (v_i, u_j), $(1 \leq i \leq m+1, 1 \leq j \leq 2)$ such that (v_1, u_1) and (v_1, u_2) are adjacent in B_m and are of degree m+1 while each of the remaining $2m$ vertices is of degree 2. We assign color 1 to the vertex (v_1, u_1), color 2 to the vertex (v_1, u_2) . The m vertices (v_i, u_2), $2 \leq i \leq m+1$ are assigned the color 1 itself. Note that none of these vertices will be adjacent to (v_1, u_1). The remaining vertices (v_i, u_1), $2 \leq i \leq m+1$ are assigned color 3, noting that these vertices will be adjacent to (v_1, u_1) with color 1. Now the only vertex in the color class 2 is (v_1, u_2). The vertex (v_1, u_1) and each of the vertices (v_i, u_2) , $2 \leq i \leq m+1$, dominates and hence power dominates the vertex (v_1, u_2), the vertex (v_1, u_2) dominates itself and each of the remaining vertices (v_i, u_1) , $2 \leq i \leq m+1$, power dominates (v_1, u_2). Hence each of the vertices of B_m power dominates the color class 2. Thus $\chi_{pd}(B_m) = 3$.

We next consider another class of graphs.

Definition 2.2

The (n, m)-lollipop graph $L_{n,m}$ is the graph [14] obtained by joining by an edge, a vertex of the complete graph K_n to an end vertex of the path P_m.

Theorem 2.2

For the Lollipop graph $L_{n,m}$, $(n \geq 2, m \geq 1)$, the power dominator chromatic number is *n i.e.* $\chi_{pd}(L_{n,m}) = n$.

Proof

Let the vertex set of the Lollipop graph $L_{n,m}$, $n \geq 2$, be $V(L_{n,m}) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_m\}$ where v_i $(1 \leq i \leq n)$ is on the complete graph K_n of the Lollipop and u_i $(1 \leq i \leq m)$ is on the path P_m. Without loss of generality, assume that the edge between K_n and P_m in the Lollipop graph, joins v_n and u_1. Since every vertex of K_n is adjacent to every other vertex, assign distinct color i to v_i, $1 \leq i \leq n$. The vertices on the path P_m of $L_{n,m}$ are colored by 1 and 2 alternatively. By the definition of power dominator coloring, each of the vertices u_j, $1 \leq j \leq m$, power dominates the color class n. Each of the vertices $v_i, 1 \leq i \leq n$, power dominates its own color class. Thus $\chi_{pd}(G) = n$.

III Power dominator Chromatic Number of Kragujevac tree

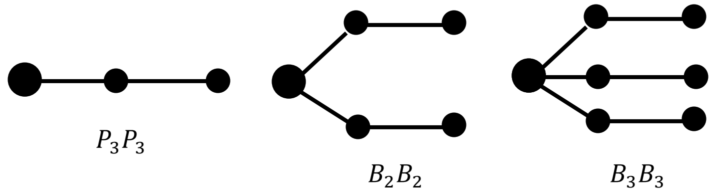

Fig. 1: Rooted trees

The path P_3 is a 3-vertex rooted tree, with the root at one of its terminal vertices. For $= 2, 3, 4, \cdots$, the rooted tree B_k *is* constructed by identifying the roots of k copies of P_3. The root of B_k is the vertex obtained by this process of identifying the roots. Examples of the rooted trees are given in Fig. 1.

Let $d \geq 2$ be an integer. Let $\beta_1, \beta_2, \dots, \beta_d$ be rooted trees as specified above *i.e.*, each of $\beta_1, \beta_2, \dots, \beta_d$ is some rooted tree B_k . A *Kragujevac tree* T is a tree [8] having a vertex of degree d, adjacent to the roots of $\beta_1, \beta_2, \dots, \beta_d$. This vertex is called the *central* vertex of T and d , the *degree* of T. The subgraphs $\beta_1, \beta_2, \dots, \beta_d$ are the *branches* of T. Note that some (or all) branches of T may be mutually isomorphic. It is clear that the branch B_k has $2k + 1$ vertices. We denote the vertices of a Kragujevac tree as follows: Pendant vertices of T are denoted by x_i , support vertex adjacent to x_i by w_i , vertex in the set $N(w_i) - \{x_i\}$ by v_i (root of a branch) and the central vertex by u .

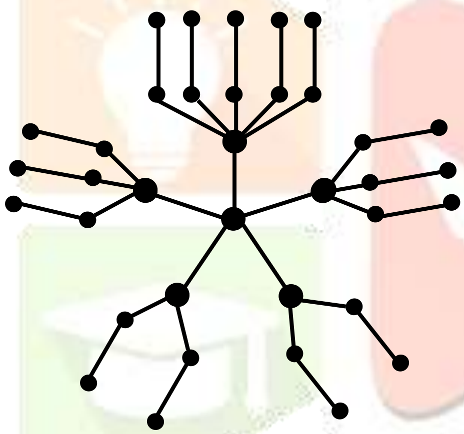

Fig. 2: A Kragujevac tree

Theorem 2.3 The power dominator chromatic number of Kragujevac tree T is $\chi_{pd}(T) = 2 + d$, where d is the degree of T.

Proof:

The Kragujevac tree T with central vertex u of degree $d \geq 2$ is adjacent to the roots of the branches $\beta_1, \beta_2, \dots, \beta_d$. We assign color 1 to the central vertex u and all the pendant vertices x_i of T and color 2 to all the support vertices w_i (adjacent to x_i) of *T*. Then the vertices v_i , the roots of the branches, are colored each by a distinct color *i*. Note that the number of such vertices is *d*, the degree of *T*. Thus we require $d + 2$ colors. Note that the vertex u dominates and hence power dominates v_i (in fact all such v_i). The vertex w_i also dominates and hence power dominates v_i while x_i power dominates v_i. Thus the vertices u, w_i, x_i power dominate the color class *i* while power dominates its own color. Hence $\chi_{pd}(T) = 2 + d$.

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