THE COMPARATIVE STUDY ON NUMERICAL SOLUTION OF IV^{th} ORDER MULTISTEP METHODS AND RK-4 FOR THE FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract : A study focuses on Runge-kutta fourth order and multistep methods of Adams Bashforth Method, Adams Moulton Method, Milne`s Methods were compared by considering first order ordinary differential equation. Moreover the effectiveness of modifiers in the RK4 method has been validated. The result of this research show that RK4 method is the most efficient method for first order ODE. The method with higher order were selected from Adams bashforth method Adams Moulton Method to serve as the RK4 respectively. The derived methods are valid and efficient. The numerical method were carried out and reveal that RK4 performed better than the Multistep method of Adams Bashforth Method, Adams Moulton Method and Milne's predictor-corrector Method.

Keywords: Initial value problem, Fourth order, single step method- Runge-Kutta, Multistep method-Adams Bashforth Method, Adams Moulton Method, Milne's predictor-corrector, Ordinary Differential Equations, ERROR analysis.

1. INTRODUCTION

Numerical methods are most widely being utilized to solve the equation arising in the fields of applied medical sciences, engineering and technology due to the advancement in the field of computational mathematics. There are several numerical methods to solve ordinary differential equation of and the multiple-step methods. Many contributions have been made in the area of numerical methods for ordinary differential equation especially in the area of the comparison of numerical methods. For comparison purpose we need to consider the problem to be solved, methods to be considered, and comparison criteria. The major factors to be considered in comparing different numerical methods are the accuracy of the numerical solution and its computation time. They further indicated that it is important to note that comparison of numerical methods is not so simple because their performances may depend on the characteristic of the problem at hand. It should also be noted that there are other factors to be considered, such as stability, versatility, proof against run-time error.

Here the method selected are the explicit Runge-Kutta method of fourth order which is a single step method and Adams Bashforth, Adams Moulton and Milne's predictor-corrector method of fourth order Multistep method. The methods selected are among the best methods available. In this method several function evaluations is performed at each step and eliminate the necessity to compute the higher derivatives. These methods can be considered for any order N. The Runge-Kutta method of order N = 4 is most popular. It is a good choice for common purposes because it is quite accurate, stable, and easy to program. Hence it is not necessary to go to a higher-order method because the increased accuracy is offset by additional computational effort. If more accuracy is required, then either a smaller step size or an adaptive method should be used. A desirable feature of a multistep method is that the local truncation error can be determined and a correction term can be included, which improves the accuracy of the answer at each step. Also it is possible if the step size is small enough to obtain an accurate value for y_{k+1}, yet large enough so that unnecessary and time-consuming calculations are eliminated. Using the combination of a predictor and corrector requires only two function evaluations of f(x, y) per step. By obtaining the predictor-corrector errors it is possible to derive Adams Bashforth, Adams Moulton and Milne's predictor-corrector method with modification formulas (yang et al., 2005). The performance of numerical methods depend on the characteristics of the ODE considered (Hull et al., 1972; Bedet et al.,1975;Butcher, 2000; Yang et al.,2005; Petzoid,2006;Clement et al.,2009; Abdul, 2013; polla, 2013; and muhammmad and Arshad, 2013). While the central activity of numerical analysis is providing accurate and efficient general purpose numerical methods and algorithms, there has always been a realization that *some problem types have distinctive features that they will need their own special theory and* techniques (Butcher, 2000).

The first problem considered in this research is a fourth-order differential equation y' = 2x + y with y(0) = 1. The second problem is $y' = y + \frac{1}{10}xy^2$, y(0) = 2 As it is shown above the problems selected have exact solutions. This is helpful to compare the approximated values with the exact values and to calculate relative errors. Hence the main purpose of this research is to compare the accuracy and computation times of Runge-Kutta, Adams Bashforth, Adams Moulton and Milne's predictor-corrector methods for fourth order ordinary differential equations.

2 PROBLEMS FORMULATION

In this section we consider two numerical method for finding the approximation solution of the initial value problems (IVP) of the first-order ordinary differential equation has the form

$$y'=f(x, y(x)), \epsilon(x_{0,}x_n) \ (1)$$
$$y(x_0) = y_0$$

Where x_0 and $y_0$are initial values for x and y respectively.

Our aim is to determine (approximately) the unknown function y(x) for x≥ x_0. We are told explicity the value of Y (x_0), namely y_0, using the given differential equation (1), we can also determine exactly the instantaneous rate of change of y at point x_0

$$y'(x_0) = f(x_0, Y(x_0))=f(x_0, y_0)$$

If the rate of y(x) were to remain f(x_0, y_0) for all point x, then y(x) would exactly

y_0+ f(x_0, y_0)(x- x_0). The rate of change of y(x) does not remain f(x_0, y_0) for all x , but it is reasonable to expect that it remain close to f(x_0, y_0) for x close to x_0 ,for small number h, and is called the step size . the numerical solution of (1) is given by a set of points {(x_n,y_n):n=0,1,2,…,n}

And each point (x_n,y_n) is an approximation to the corresponding point (x_n,y(x_n)) on the solution curve.

3. ADAMS-BASHFORTH/ADAMS-MOULTON METHOD

The Euler method,the improved Euler method ,and the Runge –Kutta method are all starting methods for the numerical solution of an initial- value problem. As we have already pointed out, a starting method y'=f(x, y) and y (x_0) = y_0 to find y_1, and in general only y'=f(x, y) and y_n to find y_{n+1}. Now, because a continuing method uses several of the preceding values y_n, y_{n-1},…, in finding y_{n+1}, it cannot be used to find the first few approximations y_1, y_2,…; they need to be found by a starting method. One begins using the continuing method after a sufficient number of y_1, y_2,…, have been found by some starting method . we consider Adams Bashforth / Adams Moulton method it is also a fourth order formulas ; hence, the rather long name of the method.

Let us refer to the Adams Bashforth / Adams Moulton method as simply the ABAM method. We will not attempt to justify the method, but will just list the formulas and explain how they are to be used.

The ABAM method can be used to approximate the value $\varphi(x_{n+1})$ of the solution φ of the initial value problems

$$y'=f(x, y),$$
$$y(x_0) = y_0,$$

at x_{n+1}=x_0 +(n+1)h, provided we have previously found approximations y_n, y_{n-1}, y_{n-2}, and y_{n-3} corresponding to the four previous points x_n, x_{n-1}, x_{n-2}, and x_{n-3}.

The method proceeds as follows: we use to determine y' at each of x_n, x_{n-1}, x_{n-2},and x_{n-3}. In particular, we set y'_n=f(x_n,y_n), y'_{n-1}=f(x_{n-1},y_{n-1}), y'_{n-2}=f(x_{n-2},y_{n-2}), and y'_{n-3}=f(x_{n-3},y_{n-3}). Using these values, we find an initial approximation y_{n+1} to $\varphi(x_{n+1})$ by the fourth order formula.

3.1 Adams Bashfoth Method of Order 4

The Adams Bashforth methods are explicit methods. The coefficients are a_{s-1} = -1 and a_{s-2} = ...= a_0 = 0,while the b_j are chosen such that the methods has order s.

The Adams Bashforth methods with s = 1,2,3,

$$y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$$

3.2 Adams Moulton method of order 4

The Adams Moulton Methods are similar to the Adams Bashforth methods in that they also a_{s-1} = -1 and a_{s-2} = ...= a_0 = 0.Again the b coefficients are chosen to obtain the highest order possible. However, the Adams Moulton methods are implicit methods. By removing the restriction that b_s = 0, an s-step Adams Moulton method can reach order a + 1, while an s-step Adams Moulton methods has only orders.

The Adams Moulton methods with s = 0,1,2,3

$$y_{n+1} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$$

4. MILNE'S PREDICTOR-CORRECTOR MRTHOD

Milne's method is a simple and reasonably accurate method of solving ordinary differential equation numerically. To solve the differential equation y'= f(x,y) by this method, first we approximate the value of y_{i+1} by predictor formula at x = x_{i+1} and then improve this value of y_{i+1} by using a corrector formula.

Milne's Predictor-corrector Method of order 4

$$y_{n+1}, p = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$
$$y_{n+1}, c = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

5. RUNGA-KUTTA METHOD(RK4)

The Runge-Kutta method is most popular because it is quite accurate, stable and easy to program. This method is distinguished by their order they agree with Taylor's series method. The fourth order Runge-Kutta method (RK4) is widely used for solving initial value problems (IVP) for ordinary differential equation (ODE).

Runge-Kutta Method of Order 4

$$\boldsymbol{k_1} = hf(x_n, y_n),$$
$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$
$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right),$$
$$k_4 = hf(x_n + h, y_n + k_3),$$
$$K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Then We set,

$$y_{n+1} = y_n + K$$

6. ACCURACY

To find which numerical method gives a more accurate approximation this study compared the errors obtained by using fourth order formulas for Runge-kutta, Adams Bashforth, Adams Moulton, and Milne's predictor-corrector methods for the two problems selected.

7. RESULTS

Comparison of exact and Error analysis

Data about exact and error analysis obtained using Runge-kutta, Adams Bashforth Method and Adams Moulton method, Milne's predictor-corrector method by varying the number of steps for the two problems. Runge-kutta fourth order has the greatest method the approximating the solution of the ordinary differential equation for all the problems considered.

Comparison of accuracy

To compare the accuracies of the Runge-kutta, Adams Bashforth, Adams Moulton, Milne's predictor-corrector methods, errors for the two problems considered. Moreover Exact and Errors are also calculated to further strengthen the comparison task. Runge-kutta fourth order method has a better accuracy the initial value problems.

8. NUMERICAL ANALYSIS

In this section we consider two numerical examples to prove which numerical methods converge faster to analytical solution. Numerical result and errors are computed and the outcomes are represented by graphically.

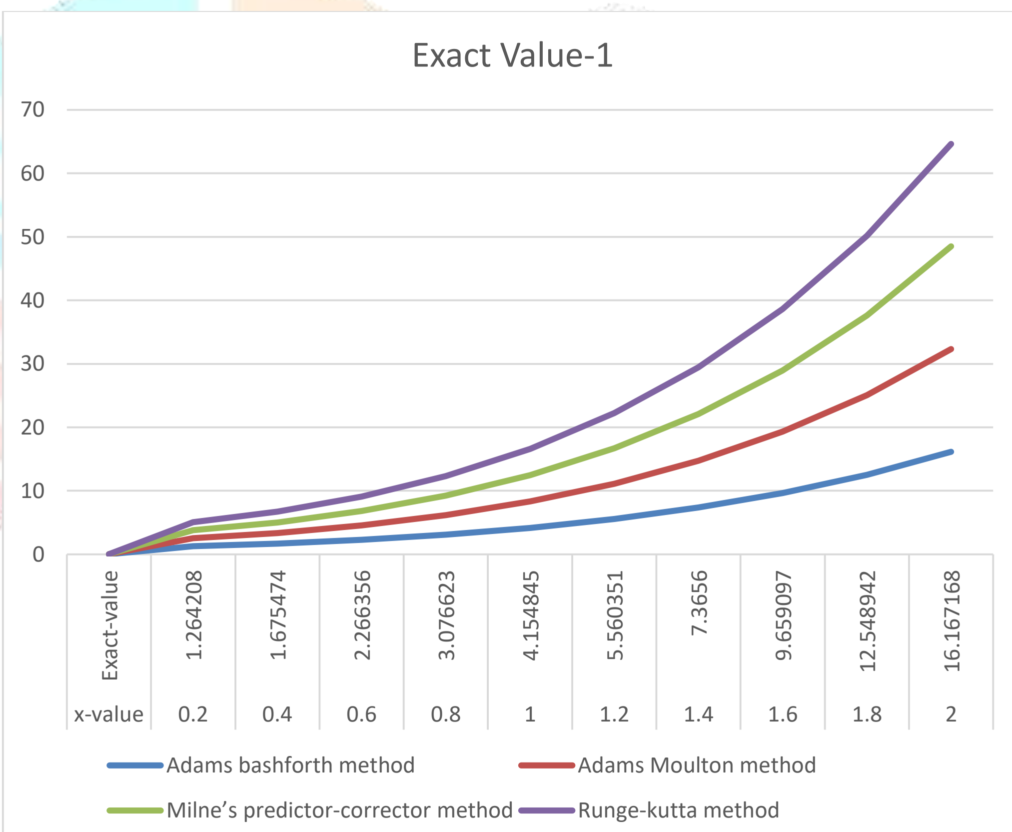
Example-1 : we consider the initial value problem y' = 2x+y , y(0) = 1 . The exact solution of the given problem is given by y = -2(x+1) + $3e^x$ The approximate results and maximum errors are obtained and shown in Tables 1 (a,b) and the graphs of the numerical solutions are displayed Figures 1 (a,b).

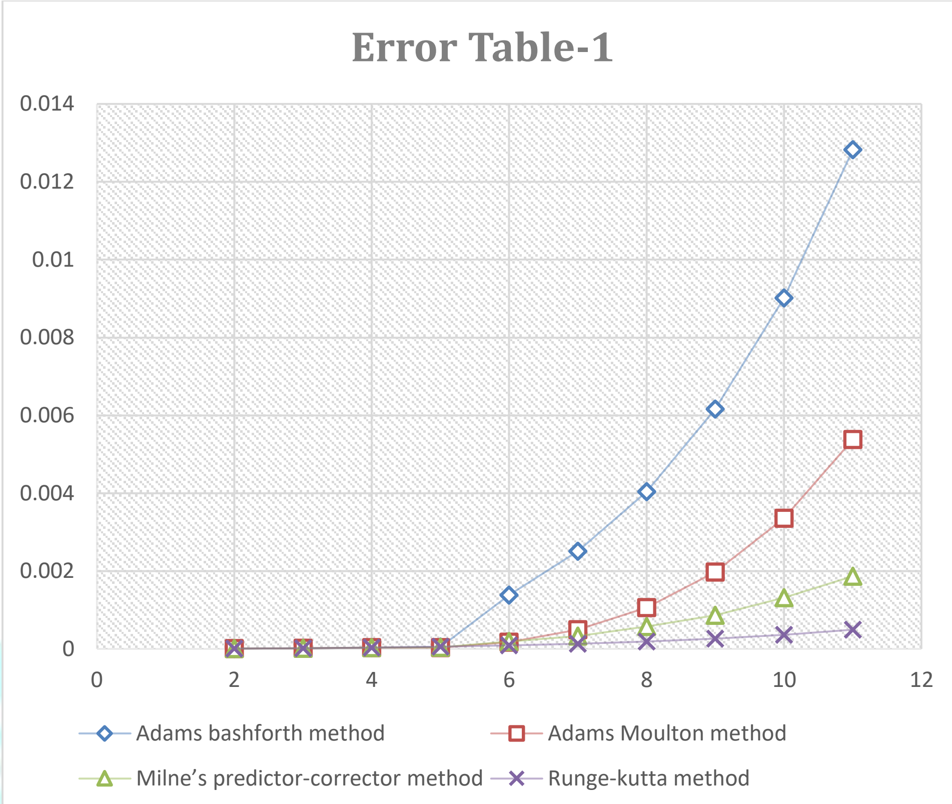
Table: 1(a): List the numerical and analytical values of y = -2(x+1) + $3e^x$ the values for each x value, Adams Bashforth, Adams Moulton, Milne's predictor-corrector, Runge-kutta methods

x-value	Exact-value	Adams bashforth method	Adams Moulton method	Milne's predictor-corrector method	Runge-kutta method
0.2	1.264208	1.264200	1.264200	1.264200	1.264200
0.4	1.675474	1.675454	1.675454	1.675454	1.675454
0.6	2.266356	2.266319	2.266319	2.266319	2.266319
0.8	3.076623	3.076079	3.076584	3.076578	3.076563
1.0	4.154845	4.153458	4.154671	4.154651	4.154753
1.2	5.560351	5.557843	5.559856	5.560019	5.560216
1.4	7.365600	7.361555	7.364538	7.365022	7.365408
1.6	9.659097	9.652938	9.657123	9.6582229	9.658829
1.8	12.548942	12.539929	12.545584	12.547623	12.548574
2.0	16.167168	16.154343	16.161789	16.165297	16.166668

Table- 1 (b) shows the errors of ABM.AMM.MP-C RK4 fourth order methods with exact method. These error values for each x are in the orders

x-value	Adams bashforth method	Adams Moulton method	Milne's predictor-corrector method	Runge-kutta method
0.2	0.000008	0.000008	0.000008	0.000008
0.4	0.000020	0.000020	0.000020	0.000020
0.6	0.000037	0.000037	0.000037	0.000037
0.8	0.0000544	0.000039	0.000045	0.000060
1.0	0.001387	0.000174	0.000194	0.000092
1.2	0.002508	0.000495	0.000332	0.000135
1.4	0.004045	0.001062	0.000578	0.000192
1.6	0.006159	0.001974	0.000868	0.000268
1.8	0.009013	0.003358	0.001319	0.000369
2.0	0.012825	0.005379	0.001871	0.000501





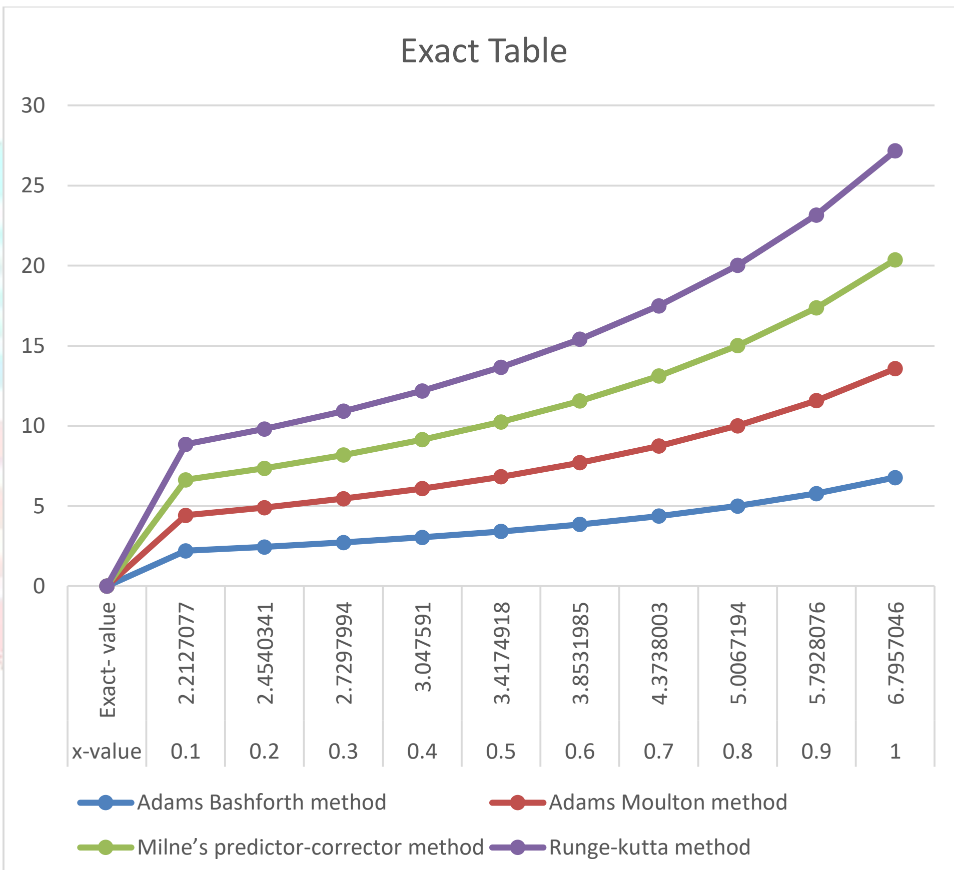
Example-2:

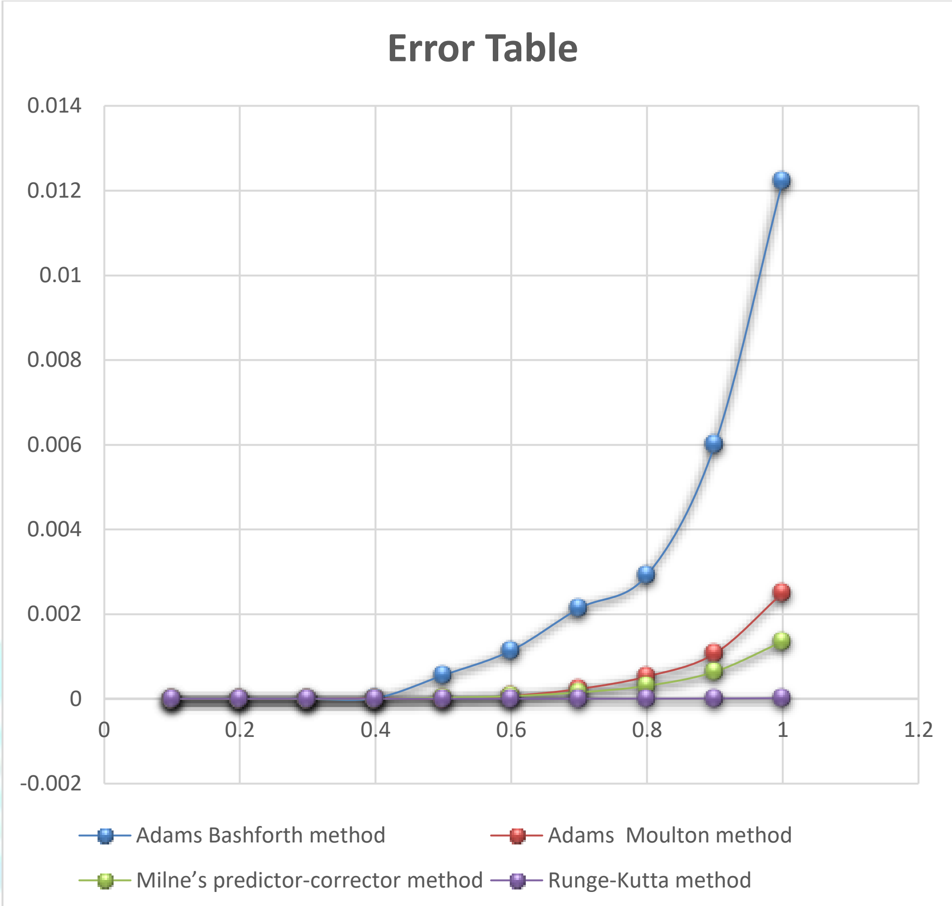
We consider the initial value problem $y' = y + \frac{1}{10}xy^2$, y(0) = 2. The exact solution of the given problem by $y = \frac{-10}{(x-1)-4e^{-x}}$, The approximate results and maximum errors are obtained and shown Tables 2(a,b) and the graphs of the numerical solutions are displayed in figures :2(a,b)

Table-2(a,b) shows the errors of ABM, AMM,MP-C,RK4 fourth order method and exact method

x-value	Exact- value	Adams Bashforth method	Adams Moulton method	Milne's predictor-corrector method	Runge-kutta method
0.1	2.2127077	2.2127076	2.2127076	2.2127076	2.2127076
0.2	2.4540341	2.4540338	2.4540338	2.4540338	2.4540338
0.3	2.7297994	2.7297988	2.7297988	2.7297988	2.7297988
0.4	3.0475910	3.0473899	3.0476016	3.0475927	3.0475900
0.5	3.4174918	3.4169309	3.4174859	3.4174608	3.4174902
0.6	3.8531985	3.8520559	3.8531261	3.8531303	3.8531960
0.7	4.3738003	4.3716624	4.3735647	4.3736451	4.3737965
0.8	5.0067194	5.0037936	5.0061905	5.0064103	5.0067131
0.9	5.7928076	5.7867989	5.7917343	5.7921651	5.7927969
1.0	6.7957046	6.7834691	6.7932030	6.7943530	6.7956847

x-value	Adams Bashforth method	Adams Moulton method	Milne's predictor-corrector method	Runge-Kutta method
0.1	0.0000001	0.0000001	0.0000001	0.0000001
0.2	0.0000003	0.0000003	0.0000003	0.0000003
0.3	0.0000006	0.0000006	0.0000006	0.0000006
0.4	0.0000201	0.0000106	0.0000017	0.0000010
0.5	0.0005609	0.0000059	0.0000310	0.0000016
0.6	0.0011426	0.0000724	0.0000682	0.0000024
0.7	0.0021379	0.0002356	0.0001552	0.0000038
0.8	0.0029258	0.0005289	0.0003091	0.0000062
0.9	0.0060087	0.0010733	0.0006425	0.0000107
1.0	0.0122355	0.0025016	0.0013516	0.0000199





Discussion of Results:

The method is of general applicability and it is the standard to which we compare the accuracy of the various other numerical methods for solving a Ordinary Differential Equation with Initial Values. We already compared the fourth orders Adams Bashforth Method, Adams Moulton Method, Milne’s predictor-corrector method and Runge-Kutta Method with Exact method in this paper.

CONCLUSION

In this paper, we have introduced a fourth order ABM, AMM, MPCM, RK4 Methods for solving ordinary differential equation. The numerical test problems have shown that the numerical solution obtained by RK4 method are good agreement with exact solution. comparing against Adams Bashforth method, Adams Moulton method, Milne’s Predictor-corrector methods.

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