A Fixed Point Theorem in Hilbert -2 Space

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Abstract: In this paper, we establish a common fixed point theorem involving commuting mapping in Hilbert-2 Space.

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1. Introduction .

The study of properties and application of fixed points of various type of contractive mappings in Hilbert-2 and Banach-2 spaces were obtained among others by Browder[1], Browder and Petryshyn[2,3],Hicks and Huffman[8], Huffman[4], Koparde and Waghmode [6], Smita Nair and Shalu Shrivastava[7]

In This paper we present a common fixed point theorem involving commutative mapping, in Hilbert -2 space.``

2. Definitions:

Definition 2.1 (Norm) : If X is a Linear space with an inner product $(\cdot , \cdot)$ then we can defined a norm on X by $\|x\| = \sqrt{(x, x)}$

Fact:

(i) $(\forall\ x \in X)$, $\|x\| \geq 0$; if and only if $\|x\| = 0$.

(ii) $(\forall\ \alpha \in C)$, $(\forall\ x \in X)$, $\|\alpha x\| = |\alpha| \|x\|$

(iii) $(\forall\ x, y \in X)$, $\|x + y\| \leq \|x\| + \|y\|$

Definition 2.2 (Cauchy Schwarz Inequality) : For all $x, y \in X$, $|(x\ ,\ y)| \leq \|x\| \cdot \|y\|$ with equality if and only if x and y are linearly dependent. Where norm is defined as above.

Definition 2.3 (Parallelogram Law) : Let X be an inner product space then $(\forall\ x, y \in X)$

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Theorem 2.1: Suppose $(X , \|\cdot\|)$ is a normed Linear space. Then norm $\|\cdot\|$ is induced by an inner product space if and only if the Parallelogram Law holds in $(X , \|\cdot\|)$.

Definition 2.4 (Continuity of Inner Product): Let be an Inner product space with induced norm $\|\cdot\|$, Then $(\cdot , \cdot) : X \times X \to C$ is continuous.

Definition 2.5 (Hilbert space) : An inner product space which is complete with respect to the norm induced by the inner product , i.e., if every Cauchy sequence is convergent , is called Hilbert space. The letter H will always denote a Hilbert space.

Example: $X = C^n$, For $x = (x_1, x_2, x_3, \ldots, x_n)$ and $y = (y_1, y_2, y_3, \ldots y_n) \in C^n$.

Then $(x, y) = \sum_{j=1}^{m} x_j \overline{y_j}$, $\|x\| = \sqrt{\sum_{j=1}^{n} |xj|^2}$ is the l^2 - norm on C^n.

Definition 2.6 (Banach Space) : A Normed linear space X is called a Banach space if it is complete, i.e., if every Cauchy sequence is convergent. We make no assumptions about the meaning of the symbol X , i.e., it need not denote a Banach space. A Hilbert space is thus a Banach space whose norm is associated with an inner product.

Theorem 2.2: Common Fixed point theorem

A pair (f, T) of self-mappings on X is said to be weakly compatible if f and T commute at their coincidence point (i.e. $fTx = Tfx$ whenever $fx = Tx$). A point $y \in X$ is called point of coincidence of two self-mappings f and T on X if there exists a point $x \in X$ such that $y = Tx = fx$.

Lemma 2.1: Let X be a non-empty set and the mappings T ; $f : X \to X$ have a Unique point of coincidence in X. If the pair (f, T) is weakly compatible, then T and f have a unique common fixed point.

Let (X, d) be a metric space, T and f be self-mappings on X, with $T(X) \subset f(X)$, and $x_0 \in X$. Choose a point x_1 in X such that $fx_1 = Tx_0$. This can be done since $T(X) \subset f(X)$. Continuing this process, having chosen $x_{1,\ldots\ldots,} x_k$, we choose x_{k+1} in X such that

$fx_{k+1} = Tx_k$; $k = 0, 1, 2, \ldots\ldots$ The sequence $\{fx_n\}$ is called a T - sequence with initial point x_0.

Lemma 2.2: [5]. Let H be a Hilbert space, then for all $x, y, z \in H$,

$$\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2$$

where $a, b, c \in [0, 1]$ and $a + b + c = 1$.

2.3 Theorem. Let E , F , T and S are four continuous self mappings of a closed subset C of a Hilbert-2 space H Satisfying

$$ES = SE,\ FT = TF,\ E(X) \subset T(X) \quad \text{And} \quad F(X) \subset S(X) \qquad \ldots\ldots\ldots\ldots(2.1)$$

$$\|Ex - Ey, a\|^2 \le a_1 \|Sx - Ex, a\|^2 \frac{[\|Ty - Fy, a\|^2 + \|Ex - Ty, a\|^2]}{\|Sx - Ty, a\|^2 + \|Ex - Ty, a\|^2} + a_2 \|Ex - Ty, a\|^2 \frac{[\|Sx - Ex, a\|^2 + \|Ty - Ey, a\|^2]}{\|Sx - Ty, a\|^2 + \|Ex - Ty, a\|^2}$$

$$+ a_3 \|Ty - Fy, a\|^2 \frac{[1 + \|Sx - Ex, a\|^2]}{1 + \|Sx - Ty, a\|^2} + a_3 \frac{[\|Ty - Fy, a\|^2 [1 + \|Sx - Ex, a\|^2]]}{1 + \|Sx - Ty, a\|^2}$$

$$+ a_4\|Sx-Ex,a\|^2\frac{[\|Ty-Fy,a\|^2]}{\|Sx-Ty,a\|^2}+a_5\|Sx-Ex,a\|^2[\|Ty-Fy,a\|^2 + a_6\|Sx-Ty,a\|^2$$

………………(2.2)

For all $x, y \in C$ with $Sx \neq Ty$

$\|Sx-Ty,a\|^2+\|Ex-Ty,a\|^2 \neq 0$ for all $a_1, a_2, a_3, a_4, a_5, a_6 \geq 0$

$a_6 < 1$ and $a_1+a_2+a_3+a_4+a_5 < 1$

Then E, F, T and S have a unique common fixed point.

Proof:

Let $x \in C$, by (1.1) there exist a point $x_1 \in C$, such that $Tx_1 = Ax_0$ and for this point x_1,we can choose a point $x_2 \in C$, such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in C such that

$y_{2n} = Tx_{2n+1} = Ex_{2n}$ and $y_{2n+1} = Sx_{2n+1} = Fx_{2n+1}$, …….(2.3)

For all $n = 0,1,2,3,……..$

From (2.2) we have

$$\|y_{2n}-y_{2n+1},a\|^2 = \|Ex_{2n}-Fx_{2n+1},a\|^2$$

$$\leq a_1\|Sx_{2n}-Ex_{2n},a\|^2\frac{[\|Tx_{2n+1}-Fx_{2n+1},a\|^2+\|Ex_{2n}-Tx_{2n+1},a\|^2]}{\|Sx_{2n}-Tx_{2n+1},a\|^2+\|Ex_{2n}-Tx_{2n+1},a\|^2}$$

$$+a_2\|Ex_{2n}-Tx_{2n+1},a\|^2\frac{[\|Sx_{2n}-Ex_{2n},a\|^2+\|Tx_{2n+1}-Fx_{2n},a\|^2]}{\|Sx_{2n}-Tx_{2n+1},a\|^2+\|Ex_{2n}-Tx_{2n+1},a\|^2}$$

$$+a_3\frac{\|Tx_{2n+1}-Fx_{2n+1},a\|^2+[1+\|Sx_{2n}-Ex_{2n},a\|^2]}{1+\|Sx_{2n}-Tx_{2n+1},a\|^2}$$

$$+a_4\frac{[\|Sx_{2n}-Ex_{2n},a\|^2+\|Tx_{2n+1}-Fx_{2n+1},a\|^2]}{\|Sx_{2n}-Tx_{2n+1},a\|^2}$$

$$+a_5[\|Sx_{2n}-Ex_{2n},a\|^2\|Tx_{2n+1}-Fx_{2n+1},a\|^2]+a_6\|Sx_{2n}-Tx_{2n+1},a\|^2$$

$$\leq a_1\|y_{2n-1}-y_{2n},a\|^2\frac{[\|y_{2n}-y_{2n+1},a\|^2+\|y_{2n}-y_{2n+1},a\|^2]}{\|y_{2n-1}-y_{2n},a\|^2+\|y_{2n}-y_{2n+1},a\|^2}$$

$$+a_2\|y_{2n}-y_{2n},a\|^2\frac{[\|y_{2n-1}-y_{2n},a\|^2+\|y_{2n}-y_{2n+1},a\|^2]}{\|y_{2n-1}-y_{2n},a\|^2+\|y_{2n}-y_{2n},a\|^2}+a_3\frac{\|y_{2n}-y_{2n+1},a\|^2+[1+\|y_{2n-1}-y_{2n},a\|^2]}{1+\|y_{2n-1}-y_{2n},a\|^2}$$

$$+a_4\frac{[\|y_{2n-1}-y_{2n},a\|^2+\|y_{2n}-y_{2n+1},a\|^2]}{\|y_{2n-1}-y_{2n},a\|^2}$$

$$+a_5[\|y_{2n-1}-y_{2n},a\|^2\|y_{2n}-y_{2n+1},a\|^2] + a_6\|y_{2n-1}-y_{2n},a\|^2$$

$$\leq (a_1+a_2+a_3+a_4)\|y_{2n}-y_{2n+1},a\|^2+(a_5+a_6)\|y_{2n-1}-y_{2n},a\|^2$$

Therefore,

$$\|y_{2n}-y_{2n+1},a\|^2 \leq \frac{(a_5+a_6)}{[1-(a_1+a_2+a_3+a_4)]}\|y_{2n-1}-y_{2n},a\|^2$$

That is $\|y_{2n}-y_{2n+1},a\|^2 \leq k\|y_{2n-1}-y_{2n},a\|^2$

where $k=\frac{(a_5+a_6)}{[1-(a_1+a_2+a_3+a_4)]}$

$$\|y_{2n}-y_{2n+1},a\|^2 \leq k\|y_{n-1}-y_n,a\|^2 \leq \ldots\ldots\ldots\ldots k^n\|y_0-y_1,a\|^2$$

For every integer $p>0$, we get

$$\|y_n-y_{n+p},a\|^2 \leq \|y_n-y_{n+1},a\|^2+\|y_{n+1}-y_{n+2},a\|^2\ldots\ldots\ldots\ldots+\|y_{n+p-1}-y_{n+p},a\|^2$$

$$\leq (1+k+k^2+\ldots..+k^{p-1})\|y_n-y_{n+p},a\|^2$$

$$\leq \frac{k^p}{1-k}\|y_n-y_{n+p},a\|^2$$

Making $n\to\infty$, we get that $\{y_n\}$ is a Cauchy sequence in C and as C is closed.

$y_n \to u \in C$

Now as $\{Fx_{2n}\}$, $\{Fx_{2n+1}\}$, $\{Tx_{2n}\}$, $\{Sx_{2n+1}\}$ are also subsequences of $\{y_n\}$ so they will also have same limit.

Now as E, F, T and S are continuous, such that

$E(S(x_n)\to Eu$, $S(E(x_n)\to Su$, $F(T(x_n)\to Fu$, $T(F(x_n)\to Tu$.

$Eu=Fu$; $Fu=Tu$. ....(2.5)

Hence from (2.1)

$$\|EEx_{2n}-Fx_{2n+1},a\|^2$$

$$\leq a_1\|SEx_{2n}-EEx_{2n},a\|^2\frac{[\|Tx_{2n+1}-Fx_{2n+1},a\|^2+\|EEx_{2n}-Tx_{2n+1},a\|^2]}{\|SEx_{2n}-Tx_{2n+1},a\|^2+\|EEx_{2n}-Tx_{2n+1},a\|^2}$$

$$+a_2\|EEx_{2n}-Tx_{2n+1},a\|^2\frac{[\|SEx_{2n}-EEx_{2n+1},a\|^2+\|Tx_{2n+1}-Fx_{2n+1},a\|^2]}{\|SEx_{2n}-Tx_{2n+1},a\|^2+\|EEx_{2n}-Tx_{2n+1},a\|^2}$$

$$+a_3\frac{\|Tx_{2n+1}-Fx_{2n+1},a\|^2+[1+\|SEx_{2n}-EEx_{2n},a\|^2]}{1+\|SEx_{2n}-Tx_{2n+1},a\|^2}$$

$$+a_4\frac{[\|SEx_{2n}-EEx_{2n},a\|^2+\|Tx_{2n+1}-Fx_{2n+1},a\|^2]}{\|SEx_{2n}-Tx_{2n+1},a\|^2}$$

$$+a_5[\|SEx_{2n}-EEx_{2n},a\|^2\|Tx_{2n-1}-Fx_{2n-1},a\|^2]+a_6\|SEx_{2n}-Tx_{2n+1},a\|^2$$

As n→∞

$$\|Eu-u,a\|^2 \le a_1\|Su-Eu,a\|^2 \frac{[\|u-u,a\|^2+\|Eu-u,a\|^2]}{\|Su-u,a\|^2+\|Eu-u,a\|^2}$$

$$+a_2\|Eu-u,a\|^2 \frac{[\|Su-Eu,a\|^2+\|u-u,a\|^2]}{\|Su-u,a\|^2+\|Eu-u,a\|^2} \quad +a_3\|u-u,a\|^2 \frac{[\|1+Su-Eu,a\|^2]}{\|1+Su-u,a\|^2}$$

$$+a_4\|u-u,a\|^2 \frac{[\|Su-Eu,a\|^2]}{\|Su-u,a\|^2} +a_5[\|Su-Eu,a\|^2\|u-u,a\|^2] +a_6[\|Su-u,a\|^2$$

Therefore $\|Eu-u,a\|^2 \le a_6\|Su-u,a\|^2 = a_6\|Eu-u,a\|^2$ as $a_6<1$

Hence $Eu=u=Su$ that is u is a fixed point of E , F , T and S .

Uniqueness : In order to prove the **uniqueness** , Let v be the another fixed point of E , F , T and S then

$$\|u-v,a\|^2 = \|Eu-Fv,a\|^2$$

$$\le a_1\|Su-Eu,a\|^2 \frac{[\|Tv-Fv,a\|^2+\|Eu-Fv,a\|^2]}{\|Su-Tv,a\|^2+\|Eu-Tv,a\|^2}$$

$$+a_2\|Eu-Tv,a\|^2 \frac{[\|Su-Eu,a\|^2+\|Tv-Fv,a\|^2]}{\|Su-Tv,a\|^2+\|Eu-Tv,a\|^2}$$

$$+a_3\|Tv-Fv,a\|^2 \frac{[\|1+Su-Eu,a\|^2]}{[1+\|Su-Tv,a\|^2} \quad +a_4\|Su-Eu,a\|^2 \frac{[\|Tv-Fv,a\|^2]}{\|Su-Tv,a\|^2}$$

$$+a_5[\|Su-Eu,a\|^2+\|Tv-Fv,a\|^2] \ +a_6\|Su-Tv,a\|^2$$

Therefore, $\|u-v,a\|^2 \le a_6\|u-v,a\|^2$ as $a_6<1 \Rightarrow u=v$

Thus u is the unique common fixed point of E , F , T and S .

This completes the proof.

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