Fixed point theorems in Partially ordered metric space

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Abstract-In this paper, we prove fixed point theorem in a partially ordered metric space.

Keywords-Coupled fixed point,partially ordered metric space, Continuous mapping.

Introduction

A number of authors generalize of the Banach contraction principle in fixed point theorems. Recently, Bhaskar and Lakshmikantham[1] Ciric and Lakshmikantham[2] [3]Sabetghadam,Masiha and Sanatpour[4] Luong and Thuan[5] have worked in this field.

Some Definitions

Definition[1]- An Element (x,y) Є XxX is said to be coupled fixed point of the mapping F:XxX→X if F(x,y)=x and F(y,x)=y.

Definition[2]- Let (X,≤) be a partially ordered set and F:XxX→X ,we say that F has the mixed monotonic property if F(x,y) is monotonic non-decreasing in x and is monotonic non-increasing in y, i.e.; for any x,y Є X.

$$x_1, x_2 \in X, x_1 \le x_2 \Rightarrow F(x_1,y) \le F(x_2,y)$$

$$\text{and } y_1, y_2 \in X, x_1 \le x_2 \Rightarrow F(x,y_1) \le F(x,y_2)$$

Theorem: Let (X,≤) be a partially set endowed with a metric d such that (X,d) is complete. Let F : XxX→X be a mapping having the mixed monotone property on X. Suppose there exist non-negative real numbers cЄ[0,1) with c<1 for each x,y,u,v ЄX we have

$$d(F(x,y), F(u,v)) \le \frac{c\ d(x,u)[1+2\{\sqrt{d(x,u)+d(u,F(u,v))}\}^2]}{[1+\{\sqrt{d(x,F(u,v))+d(u,F(x,y))}\}^2+\{\sqrt{d(x,F(x,y))+d(u,F(u,v))}\}^2}$$

Suppose either(i)F is continuous or

(ii)X has the following properties (a) if a non- decreasing sequence $\{x_n\}$ in X converges to some points $x \in X$, then $x_n \le x\ \forall\ n$ (b) if a non-increasing sequence $\{y_n\}$ in X converges to some points $y \in X$, then $y_n \ge y\ \forall\ n$.Then F has a coupled fixed point.

<u>Proof</u>:- Choose $x_0, y_0 \in X$ and set $x_1=F(x_0, y_0)$ and $y_1=F(y_0, x_0)$. In this process, We get set $x_{n+1}=F(x_n, y_n)$ and $y_{n+1}=F(y_n, x_n)$ Then by inequality we have

$$d(x_n,x_{n+1})=d(F(x_{n-1},y_{n-1}),F(x_n,y_n))$$

$$\le c\ \frac{d(x_{n-1},x_n)+[1+2\{\sqrt{d(x_{n-1},x_n)+d(x_n,F(x_n,y_n))}\}^2]}{[1+\{\sqrt{d(x_{n-1},F(x_n,y_n))+d(x_n,F(x_{n-1},y_{n-1}))}\}^2+\{\sqrt{d(x_{n-1},F(x_{n-1},y_{n-1}))+d(x_n,F(x_n,y_n))}\}^2]}$$

$$\le c\ \frac{d(x_{n-1},x_n)+[1+2\{\sqrt{d(x_{n-1},x_n)+d(x_n,x_{n+1})}\}^2]}{[1+\{\sqrt{d(x_{n-1},x_{n+1})+d(x_n,x_n)}\}^2+\{\sqrt{d(x_{n-1},x_n)+d(x_n,x_{n+1})}\}^2]}$$

$$\le c\ \frac{d(x_{n-1},x_n)+[1+2\{\sqrt{d(x_{n-1},x_{n+1})}\}^2]}{[1+\{\sqrt{d(x_{n-1},x_{n+1})}\}^2+\{\sqrt{d(x_{n-1},x_{n+1})}\}^2]}$$

$$\le c\ \frac{d(x_{n-1},x_n)+[1+2\ d(x_{n-1},x_{n+1})]}{[1+2\ d(x_{n-1},x_{n+1})]}$$

$$\le c\ \ d(x_{n-1},x_n)$$

Similarly,

$$d(y_n,y_{n+1})=d[F(y_{n-1},x_{n-1}),F(y_n,x_n)]$$

$$\le c\ \frac{d(y_{n-1},y_n)+[1+2\{\sqrt{d(y_{n-1},y_n)+d(y_n,F(y_n,x_n)}\}^2]}{[1+\{\sqrt{d(y_{n-1},F(y_n,x_n))+d(y_n,F(y_{n-1},x_{n-1}))}\}^2+\{\sqrt{d(y_{n-1},F(y_{n-1},x_{n-1}))+d(y_n,F(y_n,x_n))}\}^2]}$$

$$\le c\ \frac{d(y_{n-1},y_n)+[1+2\{\sqrt{d(y_{n-1},y_n)+d(y_n,y_{n+1})}\}^2]}{[1+\{\sqrt{d(y_{n-1},y_{n+1})+d(y_n,y_n)}\}^2+\{\sqrt{d(y_{n-1},y_n)+d(y_n,y_{n+1})}\}^2]}$$

$$\le c\ \frac{d(y_{n-1},y_n)+[1+2\{\sqrt{d(y_{n-1},y_{n+1})}\}^2]}{[1+\{\sqrt{d(y_{n-1},y_{n+1})}\}^2+\{\sqrt{d(y_{n-1},y_{n+1})}\}^2]}$$

$$\le c\ \frac{d(y_{n-1},y_n)+[1+2\ d(y_{n-1},y_{n+1})]}{[1+2\ d(y_{n-1},y_{n+1})]}$$

$$\le c\ \ d(y_{n-1},y_n) \text{ ---(i)}$$

Which implies that

$d(y_n,y_{n+1}) \leq c\, d(y_{n-1},y_n)$ --(ii)

Adding equation (i) and (ii) we have

$d_n \leq c\, d_{n-1}$ --(iii)

Let $d_n=d(x_n,x_{n+1}) + d(y_n,y_{n+1})$.In this manner we get

$d_n \leq c\, d_{n-1} \leq$ ------------$\leq c^n d_0$ .... -(iv)

If $d_0 = 0$, Then (x_0,y_0) is a coupled fixed point of F.Suppose that $d_0 \geq 0$, Then for each r Є N We obtain by the triangle inequality we have

$d(x_n,x_{n+r}) + d(y_n,y_{n+r}) \leq [d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})+$---------$+ d(x_{n+r-1},x_{n+r})]+$

$[d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2})+$---------$+ d(y_{n+r-1},y_{n+r})]$

$=[d(x_n,x_{n+1})+d(y_n,y_{n+1})]+[d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})]+$---------$+$

$[d(x_{n+r-1},x_{n+r}) + d(y_{n+r-1},y_{n+r})]$

$\leq d_n +d_{n+1} +$-----------$+ d_{n+r-1}$

$\leq c^n (1-c^r) d_0/ (1-c)$ tends to 0 as $n\to\infty$ ------------------------------------(v)

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence. Since X is a complete then there exist x, y Є X such that

$\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

Now we show that if the inequality (i) holds then (x,y) is a coupled fixed point .

We have,

$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n ,y_n) = F(\lim_{n\to\infty} x_n , \lim_{n\to\infty} y_n) = F(x,y)$

and

$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(x_n ,y_n) = F(\lim_{n\to\infty} y_n , \lim_{n\to\infty} x_n) = F(y,x)$

Therefore (x,y) is coupled fixed point of F.Suppose that the condition (a) and (b) holds. The sequence $\{x_n\} \to x$, $\{y_n\} \to y$.

$d(F(x,y),F(x_n,y_n))$

$$\leq c \frac{d(x,x_n)\,[1+2\{\sqrt{d(x,x_n)+d(x_n,F(x_n,y_n)}\}^2]}{[1+\{\sqrt{d(x,F(x_n,y_n))+d(x_n,F(x,y))}\}^2+\{\sqrt{d(x,F(x,y))+d(x_n,F(x_n,y_n))}\}^2]}$$

$$\leq c \frac{d(x,x)\,[1+2\{\sqrt{d(x,x)+d(x,x)}\}^2]}{[1+\{\sqrt{d(x,x)+d(x,x)}\}^2+\{\sqrt{d(x,x)+d(x,x)}\}^2]}$$

Letting $n\to\infty$ we have $d(F(x,y),x) \leq 0$

This shows that $F(x,y)= x$, Similarly we can show that $F(y,x)= y$.

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