JACOBI SPECTRAL - COLLOCATION METHODS FOR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS WITH SMOOTH SOLUTIONS

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Abstract : **This work is concerned with spectral Jacobi - collocation methods for Volterra integral equations of the second kind with a weakly singular of the form $(t-s)^{-\alpha}$. When the underlying solutions are sufficiently smooth, the convergence analysis was carried out in [Chen & Tang, J. Comput. Appl. Math., 233(2009), pp. 938 - 9450]; due to technical reasons the results are restricted to $0<\mu<\frac{1}{2}$. In this work, we will improve the results to the general case $0<\mu<1$ and demonstrate that the numerical errors decay exponentially in the infinity and weighted norms when the smooth solution is involved.**

Keywords: Volterra integral equations, Spectral - Collocation methods, Convergence analysis.

1. Introduction

We Consider the linear Volterra integral equations of the second kind with weakly singular kernels

$$y(t)=g(t)+\int_0^t (t-s)^{-\mu}K(t,s)y(s)ds, \quad t\in I. \qquad (1.1)$$

Where $I=[0,T]$, the function $g\in C(I)$, $y(t)$ is the unknown function, $\mu\in(0,1)$ and $K\in C(I\times I)$ with $K(t,t)\neq 0$ for $t\in I$. Several numerical methods have been proposed for (1.1) (see, for e.g., [1, 5, 17, 18]). F0r (1.1) without the singular kernel (i.e., $\mu=0$), spectral methods and the corresponding error analysis have been provided recently [19].

As the first derivative of the solution $y(t)$ behaves like $y'(t)\sim t^{-\mu}$ (see, for e.g.,[1]), it is difficult to employ high order numerical methods for solving (1.1). In [4], a Jacobi- collocation spectral method is developed for (1.1). To handle the non-smoothness of the underlying solutions, both function transformation and variable transformation are used to change the equation into a new Volterra integral equation defined on the standard interval $[-1,1]$. However, the function transformation (see also [5]) generally makes the resulting equations and approximations more complicated. We also point out a relevant recent work [8] where we consider the case with $\mu=1/2$ (the so- called Abel integral equations) and with non - smooth solutions. In [8], only coordinate transformations are employed.

On the other hand, due to the presence of the singular factor $(t-s)^{-\mu}$ in (1.1), even with the smoothness assumption of the underlying solutions there are still difficulties in establishing a framework to obtain spectral accuracy (i.e., errors decay exponentially with the increase the degree of freedom). The study of establishing the framework was carried out in [3], but due to the technical reason the convergence results were obtained for $0<\mu<\frac{1}{2}$ only. The main purpose of this work is to extend the results in [3] to more general values of μ, i.e., $0<\mu<1.$ The main technical difference between this work and [3] will be pointed out at the end of section 2. Moreover, unlike [4,8], neither coordinate nor solution transformation will be used due to the smoothness assumption of the underlying solutions.

2. Jacobi-Collocation methods

Let $\omega^{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$ be a weight function in the usual sense, for $\alpha,\beta>-1.$ As defined in [2, 7, 16], the set of Jacobi polynomials $\{J_n^{\alpha,\beta}(x)\}_{n=0}^{\infty}$ forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)-$orthogonal system, where $L^2_{\omega^{\alpha,\beta}}(-1,1)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1)=\{\upsilon:\upsilon \text{ is measurable and } \|\upsilon\|_{\omega^{\alpha,\beta}}<\infty\},$$

equipped with the norm

$$\|\upsilon\|_{\omega^{\alpha,\beta}}=\left(\int_{-1}^{1}|\upsilon(x)|^2\,\omega^{\alpha,\beta}(x)\,dx\right)^{\frac{1}{2}},$$

and the inner product

$$(u,\upsilon)_{\omega^{\alpha,\beta}}=\int_{-1}^{1}u(x)\upsilon(x)\omega^{\alpha,\beta}(x)\,dx,\quad \forall u,\upsilon\in L^2_{\omega^{\alpha,\beta}}(-1,1),$$

For a given positive integer N, we denote the collocation points by $\{x_i\}_{i=0}^{N}$, which is the set of $(N+1)$ Jacobi-Gauss points, corresponding to the weight $\omega^{-\mu,-\mu}(x).$ Let ρ_N denote the space of all polynomials of degree not exceeding N. For any $\upsilon\in C[-1,1],$ we can define the Lagrange interpolating polynomial $I_N^{\alpha,\beta}\upsilon\in\rho_N,$ satisfying

$$I_N^{\alpha,\beta}\upsilon(x_i)=\upsilon(x_i),\quad 0\le i\le N, \tag{2.1}$$

see, for e.g., [2, 7, 16]. The Lagrange interpolating polynomial can be written in the form

$$I_N^{\alpha,\beta}\upsilon(x)=\sum_{i=0}^{N}\upsilon(x_i)F_i(x),$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with $\{x_i\}_{i=0}^{N}$.

For the sake of applying the theory of orthogonal polynomials, we use the change of variable

$$t=\frac{1}{2}T(1+x),\quad s=\frac{1}{2}T(1+\tau),$$

and let

$$u(x)=y\left(\frac{T}{2}(1+x)\right),\qquad f(x)=g\left(\frac{T}{2}(1+x)\right),$$

following [3], we obtain from (1.1) that

$$u(x)=f(x)+\int_{-1}^{x}(x-\tau)^{-\mu}\tilde{K}(x,\tau)u(\tau)\,d\tau,\qquad x\in[-1,1], \tag{2.2a}$$

where

$$\tilde{K}(x,\tau)=\left(\frac{T}{2}\right)^{1-\mu}K\left(\frac{T}{2}(1+x),\frac{T}{2}(1+\tau)\right). \tag{2.2b}$$

Firstly, equation(2.2a) holds at the collocation points $\{x_i\}_{i=0}^N$ on $[-1,1]$, associated with $\omega^{-\mu,-\mu}$:

$$u(x_i)=f(x_i)+\int_{-1}^{x_i}(x_i-\tau)^{-\mu}\tilde{K}(x_i,\tau)u(\tau)d\tau,\quad 0\le i\le N. \tag{2.3}$$

In order to obtain high order accuracy for the volterra integral equations problem, the main difficulty is to compute the integral term in (2.3). We rewrite the integral term as :

$$\int_{-1}^{x_i}(x_i-\tau)^{-\mu}\tilde{K}(x_i,\tau)u(\tau)d\tau=\int_{-1}^{1}(1-\theta)^{-\mu}K_1(x_i,\tau_i(\theta))u(\tau_i(\theta))d\theta, \tag{2.4}$$

by using the following variable change

$$\tau=\tau_i(\theta)=\frac{1+x_i}{2}\theta+\frac{x_i-1}{2},\quad \theta\in[-1,1].$$

where

$$K_1(x_i,\tau_i(\theta))=\left(\frac{1+x_i}{2}\right)^{1-\mu}\tilde{K}(x_i,\tau_i(\theta)).$$

Next, using a $(N+1)-$ point Gauss quadrature formula relative to the Jacobi weight $\{\omega_i\}_{i=0}^N$, the integration term in (2.3) can be approximated by

$$\int_{-1}^{1}(1-\theta)^{-\mu}\tilde{K}(x_i,\tau_i(\theta))u(\tau_i(\theta))d\theta \approx \sum_{k=0}^{N}\tilde{K}(x_i,\tau_i(\theta_k))u(\tau_i(\theta_k))\omega_k, \tag{2.5}$$

where the set $\{\theta_k\}_{k=0}^N$ coincides with the collocation points $\{x_i\}_{i=0}^N$ is the set of $(N+1)$ Jacobi Gauss, or Jacobi Gauss - Radau, or Jacobi Gauss - Lobatto points, corresponding to the weight $\omega^{-\mu,0}$. We use u_i, $0\le i\le N$, to approximate the function value $u(x_i)$, $0\le i\le N$, and use

$$u^N(x)=\sum_{j=0}^{N}u_jF_j(x) \tag{2.6}$$

to approximate the function $u(x)$, namely $u(x_i)\approx u_i$, $u(x)\approx u^N(x)$, and

$$u(\tau_i(\theta_k))\approx\sum_{j=0}^{N}u_jF_j(\tau_i(\theta_k)),$$

then, the Jacobi-collocation method is to seek $u^N(x)$ such that $\{u_i\}_{i=0}^N$ satisfies the following collocation equations:

$$u_i=f(x_i)+\sum_{j=0}^{N}u_j\left(\sum_{k=0}^{N}K_1(x_i,\tau_i(\theta_k))F_j(\tau_i(\theta_k))\omega_k\right),\quad 0\le i\le N. \tag{2.7}$$

We close this section by pointing out the technical difference between this work and [3]. The main difference is the choice of $\{x_i\}$ in (2.3) and $\{\theta_k\}$ in (2.5). In this work, $\{x_i\}$ is associated with $\omega^{-\mu,-\mu}$ and $\{\theta_k\}$ is associated with $\omega^{-\mu,0}$, i.e., they are different ; but in [3] both are associated with $\omega^{-\mu,0}$, i.e., $x_i=\theta_i$. Note our analysis requires to use the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials (for details, see Lemma 3.3). It will be demonstrated that if $\{x_i\}$ and $\{\theta_k\}$ are chosen differently, we can make better use of appropriate Lebesgue constants. This in turn will extend our analysis to the general values of μ.

3. Some useful lemmas

Throughout the paper C will denote a generic positive constant that is independent of N but which will depend on the length T of the interval $I=[0,T]$ and on bounds for the given functions $f,\tilde{K}$ which will be defined in (2.2b), and the index μ. We first introduce some weighted Hilbert spaces. For simplicity, denote $\partial_x \upsilon(x)=\left(\frac{\partial}{\partial x}\right)\upsilon(x)$, etc. For non-negative integer m, define

$$H^m_{\omega^{\alpha,\beta}}(-1,1):=\left\{\upsilon:\partial_x^k\upsilon\in L^2_{\omega^{\alpha,\beta}}(-1,1),0\le k\le m\right\},$$

with the semi- norm and the norm as

$$|\upsilon|_{m,\omega^{\alpha,\beta}}=\left\|\partial_x^m\upsilon\right\|_{\omega^{\alpha,\beta}},\ \ \|\upsilon\|_{m,\omega^{\alpha,\beta}}=\left(\sum_{k=0}^{m}|\upsilon|^2_{k,\omega^{\alpha,\beta}}\right)^{\frac{1}{2}},$$

respectively.

To bound approximation errors of Jacobi polynomials, only some of the L^2-norms appearing on the right - hand side of above norm enter into play. Thus, it is convenient to introduce the semi - norms

$$|\upsilon|^{m;N}_{\omega^{\alpha,\beta}}:=|\upsilon|_{H^{m;N}_{\omega^{\alpha,\beta}}(-1,1)}=\left(\sum_{k=\min(m,N+1)}^{m}\left\|\partial_x^k\upsilon\right\|^2_{L^2_{w^{\alpha,\beta}}(-1,1)}\right)^{\frac{1}{2}}.$$

Lemma 3.1. ([2, 15]) for any function υ satisfying $\upsilon\in H^m_{\omega^{\alpha,\beta}}(-1,1)$, with $-1<\alpha,\beta<1$, we have

$$\left\|\upsilon-I_N^{\alpha,\beta}\upsilon\right\|_{\omega^{\alpha,\beta}}\le CN^{-m}|\upsilon|^{m;N}_{\omega^{\alpha,\beta}},$$

$$\left\|\upsilon-I_N^{\alpha,\beta}\upsilon\right\|_{1,\omega^{\alpha,\beta}}\le CN^{1-m}|\upsilon|^{m;N}_{\omega^{\alpha,\beta}},$$

for the three families of Jacobi points.

Let us define a discrete inner product. For any $u,\upsilon\in C[-1,1]$, define

$$(u,\upsilon)_N=\sum_{j=0}^{N}u(x_j)\upsilon(x_j)\omega_j. \qquad (3.1)$$

Lemma 3.2. ([2, 15]) If $\upsilon\in H^m_{\omega^{\alpha,\beta}}(-1,1)$ for some $m\ge1$ and $\phi\in\rho_N$, then for the three families of Jacobi Gauss integration we have

$$\left|(\upsilon,\phi)_{\omega^{\alpha,\beta}}-(\upsilon,\phi)_N\right|\le CN^{-m}|\upsilon|^{m;N}_{\omega^{\alpha,\beta}}\|\phi\|_{\omega^{\alpha,\beta}}.$$

From [10], we have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials.

Lemma 3.3. Let $\left\{F_j(x)\right\}_{j=0}^{N}$ be the N-th Lagrange interpolation polynomials associated with the Gauss points of the Jacobi polynomials. Then

$$\left\|I_N^{\alpha,\beta}\right\|_\infty:=\max_{x\in[-1,1]}\sum_{j=0}^{N}\left|F_j(x)\right|=\begin{cases}O(\log N),-1<\alpha,\beta\le\dfrac{-1}{2}\\ O\left(N^{\gamma+\frac{1}{2}}\right),\gamma=\max(\alpha,\beta),otherwise.\end{cases}$$

For $r\ge0$ and $k\in[0,1]$, $C^{r,k}([-1,1])$ will denote the space of functions whose r-th derivatives are Holder continuous with exponent k, endowed with the usual norm

$$\|\upsilon\|_{r,k} = \max_{0\le k\le r}\max_{x\in[-1,1]}\left|\partial_x^k \upsilon(x)\right| + \max_{0\le k\le r}\sup_{x\ne y\in[-1,1]}\frac{\left|\partial_x^k\upsilon(x)-\partial_x^k\upsilon(y)\right|}{|x-y|^k}.$$

When $k=0$, $C^{r,0}([-1,1])$ denotes the space of functions with r continuous derivatives on $[-1,1]$, which is also commonly denoted by $C^r([-1,1])$, and with norm $\|.\|_r$.

We shall make use of a result of Ragozin [12, 13], which states that, for non-negative integers r and $k\in(0,1)$, there exists a constant $C_{r,k}>0$ such that for any function $\upsilon\in C^{r,k}([-1,1])$, there exists a polynomial function $\Im_N\upsilon\in\rho_N$ such that

$$\|\upsilon-\Im_N\upsilon\|_\infty \le C_{r,k}N^{-(r+k)}\|\upsilon\|_{r,k}. \tag{3.2}$$

Actually, as stated in [12, 13], $\Im_N$ is a linear operator from $C^{r,k}([-1,1])$ into ρ_N.

We further define a linear, weakly singular integral operator M :

$$\mathrm{M}\upsilon = \int_{-1}^{x}(x-\tau)^{-\mu}\tilde{K}(x,\tau)\upsilon(\tau)\,d\tau. \tag{3.3}$$

Below we will show that M is compact as an operator from $C([-1,1])$ to $C^{0,k}([-1,1])$ provided that the index k satisfies $0<k<1-\mu$. The proof of the following lemma can be found in [4].

Lemma 3.4. Let $k\in(0,1)$ and M be defined by (3.3). Then for any function $\upsilon\in C([-1,1])$, there exists a positive constant C, which is dependent of $\|\tilde{k}\|_{0,k}$, such that

$$\frac{\left|\mathrm{M}\upsilon(x')-\mathrm{M}\upsilon(x'')\right|}{|x'-x''|^k} \le C\max_{x\in[-1,1]}|\upsilon(x)|,$$

under the assumption that $0<k<1-\mu$, for any $x', x''\in[-1,1]$ and $x'\ne x''$. This implies that

$$\|\mathrm{M}\upsilon\|_{0,k}\le C\|\upsilon\|_\infty,\quad 0<k<1-\mu.$$

where $\|.\|_\infty$ is the standard norm in $C([-1,1])$.

4. Convergence analysis

The objective of this section is to analyze the approximation scheme (2.7). Firstly, we derive the error estimate in L^∞ norm the Jacobi collocation method.

4 .1 Error estimate in L^∞

Theorem 4.1. Let u be the exact solution to the Volterra integral equation (2.2) and the approximated solution u^N be obtained by using the spectral collocation scheme (2.7) together with a polynomial interpolation (2.9). If μ associated with the weakly singular kernel satisfies $0<\mu<1$ and $u\in H^m_{\omega^{-\mu,-\mu}}(-1,1)(m\ge 1)$, then

$$\|u-u^N\|_\infty \le \begin{cases} CN^{\frac{1}{2}-m\left(|u|^{m;N}_{\omega^{-\mu,-\mu}}+N^{\frac{-1}{2}}\log NK^*\|u\|_\infty\right)}, & \frac{1}{2}\le\mu<1, \\ CN^{\frac{1}{2}-m\left(|u|^{m;N}_{\omega^{-\mu,-\mu}}+N^{-\mu}K^*\|u\|_\infty\right)}, & 0<\mu<\frac{1}{2}, \end{cases} \tag{4.1}$$

for N sufficiently large, where

$$K^* = \max_{0\le i\le N}\left|K_1\left(x_i,\tau_i(.)\right)\right|^{m;N}_{\omega^{-\mu,0}}. \qquad (4.2)$$

proof. First, we use the weighted inner product to rewrite (2.3) as

$$u(x_i) = f(x_i) + \left(K_1\left(x_i,\tau_i(.)\right), u\left(\tau_i(.)\right)\right)_{\omega^{-\mu,0}}, \quad 0\le i\le N. \qquad (4.3)$$

By using the discrete inner product (3.1), we set

$$\left(\tilde{K}\left(x_i,\tau_i(.)\right),\phi\left(\tau_i(.)\right)\right)_N = \sum_{k=0}^{N} K_1\left(x_i,\tau_i(\theta_k)\right)\phi\left(\tau_i(\theta_k)\right)w_k,$$

then, the numerical scheme (2.7) can be written as

$$u_i = f(x_i) + \left(\frac{1+x_i}{2}\right)^{1-\mu}\left(\tilde{K}\left(x_i,\tau_i(.)\right), u^N\left(\tau_i(.)\right)\right)_N, \quad 0\le i\le N, \qquad (4.4)$$

where u^N is defined by (2.6). Subtracting (4.4) from (4.3) gives the error equation:

$$u(x_i) - u_i = \left(K_1\left(x_i,\tau_i(.)\right), e\left(\tau_i(.)\right)\right)_{\omega^{-\mu,0}} + I_{i,2}$$
$$= \int_{-1}^{x_i}(x_i-\tau)^{-\mu}\tilde{K}(x_i,\tau)e(\tau)\,d\tau + I_{i,2}, \qquad (4.5)$$

for $0\le i\le N$, where $e(x) = u(x) - u^N(x)$ is the error function,

$$I_{i,2} = \left(K_1\left(x_i,\tau_i(.)\right), u^N\left(\tau_i(.)\right)\right)_{\omega^{-\mu,0}} - \left(K_1\left(x_i,\tau_i(.)\right), u^N\left(\tau_i(.)\right)\right)_N,$$

and the integral transformation (2.4) was used here. Using the integration error estimates from Jacobi- Gauss polynomials quadrature in Lemma 3.2, we have

$$\left|I_{i,2}\right| \le CN^{-m}\left\|\partial_\theta^m K_1\left(x_i,\tau_i(.)\right)\right\|^{m;N}_{\omega^{-\mu,0}}\left\|u^N\left(\tau_i(.)\right)\right\|_{\omega^{-\mu,0}}. \qquad (4.6)$$

Multiplying $F_i(x)$ on both sides of the error equation (4.5) and summing up from $i=0$ to $i=N$ yields

$$I_N^{-\mu,-\mu}u - u^N = I_N^{-\mu,-\mu}\left(\int_{-1}^{x}(x-\tau)^{-\mu}\tilde{K}(x,\tau)e(\tau)\,d\tau\right) + \sum_{i=0}^{N} I_{i,2}F_i(x).$$

Consequently,

$$e(x) = \int_{-1}^{x}(x-\tau)^{-\mu}\tilde{K}(x,\tau)e(\tau)\,d\tau + I_1 + I_2 + I_3, \qquad (4.7)$$

where

$$I_1 = u - I_N^{-\mu,-\mu}u, \qquad I_2 = \sum_{i=0}^{N} I_{i,2}F_i(x), \qquad (4.8a)$$

$$I_3 = I_N^{-\mu,-\mu}\left(\int_{-1}^{x}(x-\tau)^{-\mu}\tilde{K}(x,\tau)e(\tau)\,d\tau\right) - \int_{-1}^{x}(x-\tau)^{-\mu}\tilde{K}(x,\tau)e(\tau)\,d\tau. \qquad (4.8b)$$

It follows from the Gronwall inequality (see [4])

$$\|e\|_\infty \le C\left(\|I_1\|_\infty + \|I_2\|_\infty + \|I_3\|_\infty\right) \qquad (4.9)$$

From Sobolev inequality (see p. 490, A. 12, [2]) and Lemma 3.1, we obtained that

$$\|I_1\|_\infty = \left\|u - I_N^{-\mu,-\mu}u\right\|_\infty$$
$$\le C\left\|u - I_N^{-\mu,-\mu}u\right\|^{1/2}_{\omega^{-\mu,-\mu}}\left\|\left(u - I_N^{-\mu,-\mu}u\right)\right\|^{1/2}_{1,\omega^{-\mu,-\mu}}$$

$$\leq CN^{\frac{1}{2}-m} |u|^{m;N}_{\omega^{-\mu,-\mu}}. \qquad (4.10)$$

Next, from (4.6), we have

$$\max_{0\le i\le N}\left|I_{i,2}\right| \le CN^{-m}\max_{0\le i\le N}\left|K_1\left(x_i,\tau_i(.)\right)\right|^{m;N}_{\omega^{-\mu,0}} . \max_{0\le i\le N}\left\|u^N\left(\tau_i(.)\right)\right\|_{\omega^{-\mu,0}}$$
$$\le CN^{-m}\max_{0\le i\le N}\left|K_1\left(x_i,\tau_i(.)\right)\right|^{m;N}_{\omega^{-\mu,0}}\left\|u^N\right\|_\infty$$
$$\le CN^{-m}\max_{0\le i\le N}\left|K_1\left(x_i,\tau_i(.)\right)\right|^{m;N}_{\omega^{-\mu,0}}\left(\|e\|_\infty+\|u\|_\infty\right) \qquad (4.11)$$

Hence, by using (4.8a), (4.11) and Lemma 3.3, we have

$$\left\|I_2\right\|_\infty = \left\|\sum_{i=0}^{N} I_{i,2}F_i(x)\right\|_\infty \qquad (4.12)$$
$$\le C\max_{0\le i\le N}\left|I_{i,2}\right|\max_{x\in[-1,1]}\sum_{j=0}^{N}\left|F_j(x)\right|$$
$$\le \begin{cases} CN^{-m}\log N\max_{0\le i\le N}\left|K_1\left(x_i,\tau_i(.)\right)\right|^{m;N}_{\omega^{-\mu,0}}\left(\|e\|_\infty+\|u\|_\infty\right), \frac{1}{2}\le\mu<1, \\ CN^{\frac{1}{2}-\mu-m}\log N\max_{0\le i\le N}\left|K_1\left(x_i,\tau_i(.)\right)\right|^{m;N}_{\omega^{-\mu,0}}\left(\|e\|_\infty+\|u\|_\infty\right), 0<\mu<\frac{1}{2}, \end{cases}$$

for sufficiently large *N*. we now estimate the third term I_3. Note that

$$I_N^{-\mu,-\mu}p(x) = p(x), \quad \left(I_N^{-\mu,-\mu}-I\right)p(x) = 0, \quad \forall p(x)\in\rho_N. \qquad (4.13)$$

As the bound of $\|I_3\|_\infty$, we use the same idea as [4]. It follows from (4.13), (3.2), and Lemma 3.3 that

$$\left\|I_3\right\|_\infty = \left\|\left(I_N^{-\mu,-\mu}-I\right)Me\right\|_\infty \le \begin{cases} CN^{-k}\log N\|e\|_\infty, \frac{1}{2}\le\mu<1, \\ CN^{\frac{1}{2}-\mu-k}\|e\|_\infty, 0<\mu<\frac{1}{2}, \end{cases}$$

where in the last step we have used Lemma 3.4 under the following assumption:

$0<k<1-\mu,$ when $\frac{1}{2}\le\mu<1,$

$\frac{1}{2}-\mu<k<1-\mu,$ when $0<\mu<\frac{1}{2},$

It is clear that

$$\left\|I_3\right\|_\infty \le \frac{1}{3}\|e\|_\infty, \qquad (4.14)$$

provided that N is sufficiently large. Combining (4.9), (4.10), (4.12) and (4.14) gives the desired estimate (4.1).

4.2 Error estimate in weighted L^2 norm

To prove the error estimate in weighted L^2 norm, we need the generalized Hardy's inequality with weights (see, e.g.,[9,14]).

Lemma 4.1. For all measurable function $f\ge 0,$ the following generalized Hardy's inequality

$$\left(\int_a^b \left|(Tf)(x)\right|^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_a^b \left|f(x)\right|^p \upsilon(x)dx\right)^{\frac{1}{p}}$$

holds if and only if

$$\sup_{a<x<b}\left(\int_x^b u(t)dt\right)^{\frac{1}{q}}\left(\int_a^x \upsilon^{1-p'}(t)dt\right)^{\frac{1}{p'}} < \infty, \quad p' = \frac{p}{p-1}$$

for the case $1 < p \le q < \infty.$ Here, T is an operator of the form

$$(Tf)(x) = \int_a^x k(x,t) f(t)dt$$

with $k(x,t)$ a give kernel, u,υ weight functions, and $-\infty \le a < b \le \infty.$
from Theorem 1 in [11], we have the following weighted mean convergence result of Lagrange interpolation based at the zeros of Jacobi polynomials.

Lemma 4.2. For every bounded function $\upsilon(x),$ there exists a constant C independent of υ such that

$$\sup_N \left\|\sum_{j=0}^N \upsilon(x_j)F_j(x)\right\|_{L^2_{\omega^{\alpha,\beta}}(-1,1)} \le C\|\upsilon\|_\infty,$$

Theorem 4.2. Let u be the exact solution to the Volterra integral equation (2.2) and approximated solution u^N be obtained by using the spectral collocation scheme (2.7) together with a polynomial interpolation (2.6). Assume that $u \in H^m_{\omega^{-\mu,-\mu}}(-1,1)$ then, for N sufficiently large

$$\left\|u-u^N\right\|_{\omega-\mu,-\mu}$$

$$\le \begin{cases} CN^{-m}\left(N^{\frac{1}{2}-k}|u|^{m;N}_{\omega^{-\mu,-\mu}} + K^*\|u\|_\infty\right), \frac{1}{2} \le \mu < 1, 0 < k < 1-\mu, \\ CN^{-m}\left(|u|^{m;N}_{\omega^{-\mu,-\mu}} + K^*\|u\|_\infty\right), 0 \le \mu < \frac{1}{2}, \frac{1}{2} < k < 1-\mu, \end{cases} \tag{4.15}$$

where K^* are defined by (4.2).

Proof. It follows from the Gronwall's lemma (see [4]) and (4.7) that

$$e(x) \le C\int_{-1}^x (x-\tau)^{-\mu}\tilde{K}(x,\tau)(I_1+I_2+I_3)(\tau)d\tau + I_1+I_2+I_3$$

By the generalized Hardy's inequality Lemma 4.1, we obtain that

$$\|e\|_{\omega^{-\mu,-\mu}} \le C\left\|\int_{-1}^x (x-\tau)^{-\mu}\tilde{K}(x,\tau)(I_1+I_2+I_3)(\tau)d\tau\right\|_{\omega^{-\mu,-\mu}}$$
$$+C\left(\|I_1\|_{\omega^{-\mu,-\mu}} + \|I_2\|_{\omega^{-\mu,-\mu}} + \|I_3\|_{\omega^{-\mu,-\mu}}\right)$$
$$\le C\left(\|I_1\|_{\omega^{-\mu,-\mu}} + \|I_2\|_{\omega^{-\mu,-\mu}} + \|I_3\|_{\omega^{-\mu,-\mu}}\right).$$

Now, using Lemma 3.1, we obtain that

$$\|I_1\|_{\omega^{-\mu,-\mu}} = \left\|u - I_N^{-\mu,-\mu}u\right\|_{\omega^{-\mu,-\mu}} \le CN^{-m}|u|^{m;N}_{\omega^{-\mu,-\mu}}. \tag{4.16}$$

Using Lemma 4.2 and (4.11) gives

$$\left\|I_2\right\|_{\omega^{-\mu,-\mu}}=\left\|\sum_{i=0}^{N} I_{i,2}F_i(x)\right\|_{\omega^{-\mu,-\mu}}\le C\max_{0\le i\le N}\left|I_{i,2}\right|$$
$$\le CN^{-m}K^*\left(\|e\|_\infty+\|u\|_\infty\right).$$

Finally, for the bound of $\left\|I_3\right\|_\infty$, we use the same idea as [4]. It follows from (4.13), Lemma 4.2, and (3.2) that

$$\left\|I_3\right\|_{\omega^{-\mu,-\mu}}=\left\|\left(I_N^{-\mu,-\mu}-I\right)Me\right\|_{\omega^{-\mu,-\mu}}\le CN^{-k}\|e\|_\infty, \qquad (4.17)$$

where in the last step we used Lemma 3.4 for any $k\in(0,1-\mu)$. By the convergence result in Theorem 4.1, we obtain that

$$\left\|I_3\right\|_{\omega^{-\mu,-\mu}}$$
$$\le\begin{cases} CN^{\frac{1}{2}-m-k}\left(|u|_{\omega^{-\mu,-\mu}}^{m;N}+N^{\frac{-1}{2}}\log NK^*\|u\|_\infty\right), \dfrac{1}{2}\le\mu<1,\\ CN^{\frac{1}{2}-m-k}\left(|u|_{\omega^{-\mu,-\mu}}^{m;N}+N^{-\mu}K^*\|u\|_\infty\right), 0\le\mu<\dfrac{1}{2}, \end{cases} \qquad (4.18)$$

for sufficiently large and for any $k\in(0,1-\mu)$. The desired estimate (4.17) is obtained by combining (4.16), (4.17), and (4.18).

5. Conclusion

In this work, we have considered the Volterra integral equation (1.1) with the assumption that the exact solutions of (1.1) are smooth. We point out that this case may occur when the source function g in (1.1) is non - smooth ; see, for e.g., Theorem 6.1.11 in [1]. In this case, the Jacobi - collocation spectral method can be applied directly, which leads to spectral accuracy without using any coordinate or function transformations. we close this work by noting that the Jacobi weighted Besov / Sobolev may be a natural tool for polynomial approximations. In [6, 7], it was demonstrated that Jacobi - weighted Besov and Sobolev spaces are the most appropriate tools for obtaining optimal upper and lower bounds when dealing with weakly singular problems, particularly for those with certain singularity at the end - points. This approach may be useful for analyzing the direct Jacobi - collocation spectral approach outlined in this work . It remains to be a future research topic.

6. **References**

[1]. A. Kufner, L. E. persson , Weighted Inequalities of Hardy Type, World Scientific, New York, 2003.
[2]. B. Guo, L. Wang, Jacobi interpolation approximations and their applications to singular differential equations, Adv. Comput. Math. 14(2001)227 - 276.
[3]. B. Q. Guo, Approximation theory for the $p-$ version of the finite element method in three dimensions, part I: Approximabilities of singular functions in the framework of the Jacob - weighted Besov and Sobolev spaces, SIAMJ. Numer. Anal., 44 (2006), 246 - 269.
[4]. C. Canuto, M.Y. Hussaini, A. Quarteroni, T. A. Zang, Spectral Methods Fundamentals in Single Domains, Springer - Verlag, 2006.
[5]. D. L. Ragozin, Polynomial approximation on compact manifolds and homogeneous spaces, Trans. Amer. Math. Soc. 150(1970) 41 - 53.
[6]. D. L. Ragozin, Constructive polynomial approximation on spheres and projective spaces, Trans. Amer. Math. Soc. 162(1971) 157 - 170.
[7]. G. Mastroianni and D. Occorsio, Optimal systems of nodes for Lagrange interpolation on bounded intervals. A survey, J. Comput. Appl. Math. 134(2001)325 - 341.
[8] H. Brunner , Collocation Methods for Volterra Integral and Related Functional Equations Methods, Cambridge University press, 2004.
[9]. J. Shen and T. Tang, Spectral and High - order Methods with Applications, Science Press, Beijing, 2006.
[10]. J. Shen, T. Tang and L. - L. Wang, Spectral Methods: Algorithms, Analysis and Applications, Springer Series in Computational Mathematics, Springer, New York, 2011.
[11]. P. Nevai, Mean convergence of Lagrange interpolation. III, Trans. Amer. Math. Soc.

282 (1984) 669 - 698. MR 85c: 41009.

[12]. S. G. Samko, R.P. Cardoso, Sonine integral equations of the first kind in $L_p(0,b)$, Fract. Calc. Appl. Anal., 6 (2003) 235 - 258.

[13]. T. Tang, Super convergence of numerical solutions to weakly singular Volterra integro-differential equations, Numer. Math. 61 (1992) 373 - 382.

[14]. T. Tang, A note on collocation methods for Volterra integro - differential equations, with weakly singular kernels, IMA J. Numer. Anal. 13 (1993) 93 - 99.

[15]. T. Diogo, S. Mckee, T. Tang, Collocation methods for second - kind Volterra integral equations with weakly singular kernels, Proc. Roy. Soc. Edinburgh 124A (1994) 199 - 210.

[16]. T. Tang, Xiang Xu and J. Cheng, on spectral methods for Volterra type integral equations and the convergence analysis, J. Comput. Math. 26(2008) 825 - 837.

[17]. X. J. Li and T. Tang, convergence analysis of Jacobi spectral collocation methods for Abel-Volterra integral equations of second kind, Front. Math. China, 7 (2012), 69 - 84.

[18]. Y. Chen and T. Tang, Spectral methods for weakly singular Volterra integral equations with smooth solutions, J. Comput. Appl. Math., 233 (2009), 938 - 950.

[19]. Y. Chen, T. Tang, Convergence analysis of the Jacobi spectral - collocation methods for Volterra integral equations with a weakly singular kernel, Math. Comp., 79(2010), 147 - 167.