ON THE STABILITY OF SECOND ORDER LINEAR DIFFERENCE AND LINEAR FUNCTIONAL EQUATIONS

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ABSTRACT:

In this work, On the stability of second order linear difference and linear functional equations of the form: $x_{n+2} + \gamma x_{n+1} + \delta x_n = 0,$

$x_{n+2} + \gamma x_{n+1} + \delta x_n = p_n$

and

$$x_{n+2} - \gamma_n x_{n+1} + \delta_n x_n = p_n$$

are studied, where $\gamma, \delta \in \mathbb{R}$ and p_n, γ_n, δ_n are sequence of reals.

1. INTRODUCTION

On the stability problem for functional equations was replaced by stability of differential equations. The differential equation

$$r_n(t)x^{(n)}(t) + r_{n-1}(t)x^{(n-1)}(t) + \cdots + r_1(t)x'(t) + r_0(t)x(t) + h(t) = 0$$

has the stability, if for given $\epsilon > 0$, I be an interval and for any function g satisfying the differential inequality

$$|r_n(t)x^{(n)}(t) + r_{n-1}(t)x^{(n-1)}(t) + \cdots + r_1(t)x'(t) + r_0(t)x(t) + h(t)| = \epsilon ,$$

then there exists a solution $g_0(t)$ of the above equation such that

$$|g(t) - g_0(t)| \leq L(\epsilon) \quad \text{and} \quad \lim_{\epsilon \to 0} L(\epsilon) = 0, t \in I$$

We have discussed on the stability of second order linear differential and linear functional equations of the form:

$$x'' + r\,x' + sx = 0 \quad \text{----------} \quad (1.1)$$

and

$$x'' + r\,x' + sx = g(t) \quad \text{----------} \quad (1.2)$$

Where $a, b \in \mathbb{R}$. The objective of this work is to study the stability of discrete analogue of the equations (1.1) and (1.2) as

$$x_{n+2} + \gamma x_{n+1} + \delta x_n = 0 \text{ ----------} \ (1.3)$$

and

$$x_{n+2} + \gamma x_{n+1} + \delta x_n = p_n \text{----------}(1.4)$$

Where $\gamma, \delta \in \mathbb{R}$ and p_n is a sequence of reals. Also, an effort is made to study on the stability of

$x_{n+2} + \gamma_n x_{n+1} + \delta_n x_n = p_n$----------(1.5)

DEFINITION: 1.1

The difference equation

$$r_k(n)x(n+k) + r_{k-1}(n)x(n+k-1) + \cdots + r_1(n)x(n+1) + r_0(n)x(n) + h(n) = 0$$

has the stability, if for given $\epsilon > 0$, I be an open interval and for any function g satisfying the inequality

$|r_k(n)x(n+k) + r_{k-1}(n)x(n+k-1) + \cdots + r_1(n)x(n+1) + r_0(n)x(n) + h(n)| \leq \epsilon,$

Then there exists a solution g_0 of the above difference equation such that

$|g(n) - g_0(n)| \leq L(\epsilon)$ and

$$\lim_{\epsilon \to 0} L(\epsilon) = 0,$$

for

$n \in I \subset N(0) = \{0,1,2,3, \dots\}.$

DEFINITION: 1.2

We say that (1.4) has the stability if there exists a constant L >0 with the property: for every $\epsilon > 0$, x_n, p_n defined for $n \in (r, s+1)$, $0 < r < s < \infty$,if

$|x_{n+2} + \gamma x_{n+1} + \delta x_n - p_n| \leq \epsilon$,------ (1.6)

Then there exists some $z_n, n \in (r, s+1)$ satisfyng

$$z_{n+2} + \gamma z_{n+1} + \delta z_n = p_n$$

Such that $|x_n - z_n| < L\epsilon$. Let L be a Hyers –Ulam stability constant for (1.4).

2. STABILITY RESULTS FOR $x_{n+2} + \gamma x_{n+1} + \delta x_n = 0$ and $x_{n+2} + \gamma x_{n+1} + \delta x_n = p_n$

Now, in this section deals with the stability of $x_{n+2} + \gamma x_{n+1} + \delta x_n = 0$ and $x_{n+2} + \gamma x_{n+1} + \delta x_n = p_n$.

THEOREM: 2.1

Assume that the characteristic equation $m^2 + \gamma m + \delta = 0$ have two different positive roots. Then (1.3) has the stability.

Proof:

Let $\epsilon > 0$ and $x_n, n \in (r, s+1)$ be a solution of (1.3) satisfying the property

$|x_{n+2} + \gamma x_{n+1} + \delta x_n| \leq \epsilon.$

Let λ and μ be the positive roots of the characteristic equation. For $n \in (r, s+1)$, define $f_n = x_{n+1} - \lambda x_n$. Then

$$f_{n+1} = x_{n+2} - \lambda x_{n+1}$$

and hence

$|f_{n+1} - \mu f_n| = |x_{n+2} - \lambda x_{n+1} - \mu x_{n+1} + \lambda \mu x_n|$

$= |x_{n+2} - (\lambda + \mu) x_{n+1} + \lambda \mu x_n|$
$= |x_{n+2} + \gamma x_{n+1} + \delta x_n| \leq \epsilon.$

Equivalently, f_n satisfies the relation

$-\epsilon \leq f_{n+1} - \mu f_n \leq \epsilon$ ----------(2.1).

Upon the choice of λ and μ, We have four possibilities.

i) $\lambda > 1, \mu > 1$; *ii)* $\lambda \leq 1, \mu \leq 1$; *iii)* $\lambda > 1, \mu \leq 1$; *iv)* $\lambda \leq 1, \mu > 1$

Consider case i)

Then (2.1) can be viewed as

$-\epsilon\mu^{-(n+1)} \leq \mu^{-(n+1)}[f_{n+1} - \mu f_n] \leq \epsilon\mu^{-(n+1)},$

i.e) $-\epsilon\mu^{-(n+1)} \leq \Delta(\mu^{-n} f_n) \leq \epsilon\mu^{-(n+1)}$---------- (2.2)

Therefore for $n \in (r, s+1)$, it follows that

$$-\epsilon\sum_{j=n}^{s}\mu^{-(j+1)} \le \sum_{j=n}^{s}\Delta(\mu^{-j}f_j) \le \epsilon\sum_{j=n}^{s}\mu^{-(j+1)}$$

Which is implies that

$$\frac{-\epsilon\mu^{-n}}{\mu-1} \le \mu^{-(s+1)}f_{s+1}-\mu^{-n}f_n \le \frac{\epsilon\mu^{-n}}{\mu-1}.$$

Consequently,

$$-\epsilon_1 \le \mu^{-(s-n+1)}f_{s+1}-f_n \le \epsilon_1,$$

Where $\epsilon_1 = \frac{\epsilon}{\mu-1}$.

Let $z_n = \mu^{-(s-n+1)} f_{s+1}$.

Then $z_{n+1}-\mu z_n = 0$. Now, $|f_n - z_n| \le \epsilon_1$ implies that

$$-\epsilon_1 \le x_{n+1}-\lambda x_n - z_n \le \epsilon_1$$

and hence

$$-\epsilon_1\lambda^{-(n+1)} \le \lambda^{-(n+1)}[x_{n+1}-\lambda x_n - z_n] \le \epsilon_1\lambda^{-(n+1)}.$$

Proceeding as above, we obtain

$$-\epsilon_1\frac{\lambda^{-n}}{\lambda-1} \le \lambda^{-(s+1)}x_{s+1}-\lambda^{-n}x_n - \sum_{j=n}^{s}\lambda^{-(j+1)}z_j \le \epsilon_1\frac{\lambda^{-n}}{\lambda-1},$$

(i.e),

$$\frac{-\epsilon_1}{\lambda-1} \le \lambda^{-(s-n+1)}x_{s+1}-x_n - \lambda^{n}\sum_{j=n}^{s}\lambda^{-(j+1)}z_j \le \frac{\epsilon_1}{\lambda-1}.$$

Denote that,

$$u_n = \lambda^{-(s-n+1)}x_{s+1}-\sum_{j=n}^{s}\lambda^{-(j-n+1)}z_j,$$

Then $|u_n - x_n| \le \frac{\epsilon_1}{\lambda-1} = \frac{\epsilon}{(\lambda-1)(\mu-1)}$.

It is easy to verify that $u_{n+1} = \lambda u_n + z_n$ and hence

$$u_{n+2}-\lambda u_{n+1} = z_{n+1} = \mu z_n = \mu[u_{n+1}-\lambda u_n]$$

implies that

$$u_{n+2}+\gamma u_{n+1}+\delta u_n = 0.$$

Consequently, (1.3) has the stability with the stability constant

$$L = \frac{1}{(\lambda-1)(\mu-1)}.$$

Next, we consider Case(ii).

Assume that there exist positive integers M,N>0 such that $\mu M > 1$ and $\lambda N > 1$. Using the same type of argument as in case (i), we get the equation (2.2) and hence

$$-\epsilon\sum_{j=n}^{s}(\mu M)^{-(j+1)}M^{j+1} \le \sum_{j=n}^{s}\Delta(\mu^{-j}f_j) \le \epsilon\sum_{j=1}^{s}(\mu M)^{-(j+1)}M^{j+1}$$

----------(2.3)

Let $M^* = \max\{M^{n+1}, M^{n+2}, \ldots M^{s+1}\}, r \le n < s+1$.

Then (2.3) becomes

$$-\epsilon M^*\sum_{j=n}^{s}(\mu M)^{-(j+1)} \le \mu^{-(s+1)}f_{s+1} - \mu^{-n}f_n \le \epsilon M^*\sum_{j=n}^{s}(\mu M)^{-(j+1)},$$

(i.e),

$$M^*\frac{-\epsilon(\mu M)^{-n}}{(\mu M-1)} \le \mu^{-(s+1)}f_{s+1}-\mu^{-n}f_n \le M^*\frac{\epsilon(\mu M)^{-n}}{(\mu M-1)}.$$

Consequently,

$$\frac{-\epsilon M^*}{(\mu M-1)M^r} \leq \mu^{-(s-n+1)}f_{s+1} - f_n \leq \frac{-\epsilon M^*}{(\mu M-1)M^r}$$

The rest of the proof follows from Case (i).we note that the stability constant is given by $K = \frac{\epsilon M^* N^*}{(\mu M-1)(\lambda N-1)MN)^r}$,

Where $N^* = \max\{N^{n+1}, N^{n+2}, \ldots N^{s+1}\}, r \leq n < s+1$.Cases (iii)and (iv) follow from Cases (i) and (ii).

Hence the proof.

THEOREM 2.2:

Assume that the characteristic equation $m^2 + \gamma m + \delta = 0$ have two different positive roots. Furthermore, assume that (1.6) holds .Then (1.4) has the Hyers-ulam stability.

Proof:

Proceeding as in the proof of theorem 2.1, we obtain

$$|f_{n+1} - \mu f_n - p_n| = |x_{n+2} - \lambda x_{n+1} - \mu x_{n+1} + \lambda\mu x_n - p_n|$$

$$= |x_{n+2} - (\lambda + \mu)\, x_{n+1} + \lambda\mu x_n - p_n|$$

$$= |x_{n+2} + \gamma x_{n+1} + \delta x_n - p_n| \leq \epsilon.$$

Equivalently, f_n satisfies the relation

$$-\epsilon \leq f_{n+1} - \mu f_n - p_n \leq \epsilon.$$

Similar to Theorem 2.1,we have four possibilities upon the choices of λ and μ.

We consider Case (i) only.

And hence, similar to the equation (2.2),

We have

$$-\epsilon\mu^{-(n+1)} \leq \Delta(\mu^{-n} f_n) - \mu^{-(n+1)} p_n \leq \epsilon\mu^{-(n+1)}$$

and let

$$z_n = \mu^{-(s-n+1)} f_{s+1} - \mu^n \sum_{j=n}^{s} \mu^{-(j+1)} p_j$$

.

Therefore, z_n satisfies $z_{n+1} - \mu z_n - p_n = 0$, and $|f_n - z_n| \leq \epsilon_1$.

Using the same type of argument as in Theorem 2.1, We can show that there exists

$$u_n = \lambda^{-(s-n+1)} x_{s+1} - \sum_{j=n}^{s} \lambda^{-(j-n+1)} z_j$$

such that $|u_n - x_n| \leq \frac{\epsilon}{(\lambda-1)(\mu-1)}$ and u_n satisfies

$$u_{n+2} + \gamma u_{n+1} + \delta u_n - p_n = 0.$$

Hence the proof.

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