# SOLVING LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENT BY FACTORIZATION

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**Abstract**

This paper is concerned with the impulsive response method for solving linear constant – coefficient ordinary differential equations. In particular we avoid the use of distribution theory, as well as of the other more advanced approaches linear systems, laplace transform, the general theory of linear equations with variable coefficients and the variation of constants method.

**Keywords**

Linear differential equations, factorization, homogenous equation, continuous function.

**Introduction**

In introductory courses on differential equations, the treatment of second or higher order non-homogeneous equations is usually limited to illustrating the method of undetermined coefficients. It is well known that the impulsive response method gives an explicit formula for a particular solution in the more general case in which the forcing term is an arbitrary continuous function. An alternative approach, which is sometimes used consists in developing the theory of linear systems first, considering then linear equations of order $n$ as a particular case of this theory.

**First-order equations**

Consider the first- order linear differential equation

$$y' + by = f(x), \qquad \ldots\ldots (1)$$

Where $y' = \frac{dy}{dx}$, $b$ is a real constant, and the forcing term $g$ is a continuous function in an interval $I \subset \mathbb{R}$. It is well known that the general solution of (1) is given by

$$y(x) = e^{-bx} \int e^{bx} g(x)dx \qquad \ldots\ldots (2)$$

Where $\int e^{bx} g(x)dx$ denotes the set of all primitives of the functions $e^{bx} g(x)$ in the interval $I$.

Suppose that $0 \epsilon I$, and consider the integral function $\int_o^x e^{bt} f(t)dt$. By the Fundamental Theorem of Calculus, this is the primitive of $e^{ax} f(x)$ that vanishes at $0$. The theorem of the additive constant for primitives implies that

$$\int e^{bx} g(x)dx = \int_o^x e^{bt} g(t)dt + \kappa \quad (\kappa \epsilon \mathbb{R})$$

And we can rewrite (2) in the form

$$y(x) = e^{bt} g(t)dt + ke^{-ax}$$

$$= \int_0^x e^{-a(x-t)} f(t)dt + ke^{-ax}$$

$$= \int_0^x h(x-t)g(t)dt + kh(x), \qquad \ldots\ldots (3)$$

Where $h(x) = e^{-bx}$.The function h is called the $impulsive\ response$ of the differential equation $y' + by = 0$. It is the unique solution of the initial value problem

$$\begin{cases} y' + by = 0 \\ y(0) = 1. \end{cases}$$

Formula (3) illustrates a well- known result in the theory of liner differential equations. Namely, the general solution of (2.1) is the sum of the general solution of the associated homogeneous equation $y' + ay = 0$ and of any particular solution (1). In (3)the function

$$y_p(x) = \int_0^x h(x-t)g(t) \qquad \ldots\ldots (4)$$

is the particular solution of (1) that vanishes at $x = 0$.

If $x_0$is any point of $I$ ,it is easy to verify that

$$y(x) = \int_{x_0}^x h(x-t)g(t)dt + y_0\, h(x-x_0) \qquad (x\epsilon I)$$

is the unique solution of (1) in the interval I that satisfies

$$y(x_0) = y_0 (y_0 \epsilon \mathbb{R}).$$

We shall now see that the formula (4) gives a particular solution of the non-homogenous equation also in the case of second-order liner constant-coefficient differential equations, by suitably defining the impulsive response $g$.

**Second-order equations**

Consider the second-order non-homogeneous linear differential equation

$$y'' + ay' + by = g(x) \qquad \ldots\ldots.. (5)$$

Where $y'' = \frac{d^2y}{dx^2}, a, b\epsilon\mathbb{R}$, and the forcing term $g : I \to \mathbb{R}$ is a continous function in the interval $I \subset \mathbb{R}$,i.e.$g\epsilon c^0(I)$. For $g = 0$ we get the associated homogenous equation

$$y'' + ay' + by = 0 \qquad \ldots\ldots.. (6)$$

We will write (5) and (6) in the operator form as $Ly = g(x)$ and $Ly = 0$,

where $L$ is the $linear\ second-order$ *differential operator with constant coefficients* defined by

$$Ly = y'' + ay' + by,$$

for any function $y$at least twice differentiable.

Denoting by$\frac{d}{dx}$ the differentiation operator, we have

$$L = \left(\frac{d}{dx}\right) + a\frac{d}{dx} + b. \qquad \ldots\ldots. (7)$$

$L$ defines a map $c^2(\mathbb{R}) \to c^0(\mathbb{R})$ that to each function $y$at least twice differentiable over $\mathbb{R}$ with continuous second derivative associates the continuous function $Ly$.The fundamental property of $L$is its linearity, that is,

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2$$

$$\forall c_1, c_2 \in \mathbb{R},\ \forall y_{1,}y_2 \in c^2(\mathbb{R}).$$

$$V = \{y \in c^2(\mathbb{R}) : Ly = 0$$

is a vector space over $\mathbb{R}$. We shall see that this vector space has dimension two, and that the solutions of (7) are defined in fact on the whole of $\mathbb{R}$ and are of class $c^\infty$there.

Second, if $y_1$ and $y_2$ are two solutions of (6) (in a interval $I' \subset I$), then their difference $y_1 - y_2$ solves (7).It follows that if we know a particular solution $y_p$ of the non-homogenous equation (in an interval $I'$),then any other solution of (6) in $I'$ is given by $y_p + y_{h,}$

The fact that $L$ has constant coefficients (i.e. $a$ and $b$ in (7) are constants ) Allows one to find explicit formulas for the solutions of (6 ) and (7). To this end, it is useful to consider complex- valued functions $y : \mathbb{R} \to \mathbb{C}$. If $y = y_1 + iy_2$(with $y_{1,}y_2 :: \mathbb{R} \to \mathbb{R}$) is such a function, the derivative $y'$ may be defined by linearity as $y' = y_1' + iy_2'$. It follows that

$L(y_1 + iy_2) = Ly_1 + iy_2$. In a similar way one defines the integral of $y$ as

$$\int y(x)dx = \int y_1(x)dx + i\int y_2(x)dx,$$

$$\int_c^d y(x)dx = y_1(x)dx + i\int_c^d y_2(x)dx.$$

It is then easy to verify that

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x} \quad \forall\lambda \in \mathbb{C}.$$

It follows that the complex exponential $e^{\lambda x}$ is a solution of (6) if and only if $\lambda$ is a root of the characteristic polynomial

$$p(\lambda) = \lambda^2 + a\lambda + b.$$

Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the roots of $p(\lambda)$, so that

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2).$$

The operator $L$ factors in a similar way as a product (composition ) of first –order differential operator

$$L = \left(\frac{d}{dx} - \lambda_1\right)\left(\frac{d}{dx} - \lambda_2\right). \quad \ldots.(8)$$

Indeed we have

$$\left(\frac{d}{dx} - \lambda_1\right)\left(\frac{d}{dx} - \lambda_2\right)y = \left(\frac{d}{dx} - \lambda_1\right)(y' - \lambda_2 y)$$

$$= y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y,$$

Which coincides with $Ly$ since $\lambda_1 + \lambda_2 = -a$ and $\lambda_1\lambda_2 = b$. Note that in (8) the order with which the two factors are composed is unimportant. In other words, the two operators$\left(\frac{d}{dx} - \lambda_1\right)$and $\left(\frac{d}{dx} - \lambda_2\right)$ commute

$$\left(\frac{d}{dx} - \lambda_1\right)\left(\frac{d}{dx} - \lambda_2\right) = \left(\frac{d}{dx} - \lambda_2\right)\left(\frac{d}{dx} - \lambda_1\right).$$

The idea is now to use (8) to reduce the problem to first-order differential equations. It is useful to consider linear differential equations with complex coefficients, whose solutions will be, in general, complex-valued. For example, the first-order homogenous equation $y' - \lambda y = 0$ with $\lambda \in \mathbb{C}$ has the general solution

$$y(x) = ke^{\lambda x} \qquad (k \in \mathbb{C}).$$

(Indeed if $y' = \lambda y$,then $\frac{d}{dx}\left(y(x)e^{-\lambda x}\right) = 0$, whence $y(x)e^{-\lambda x} = k$.) The first-order non-homogenous equation

$$y' - \lambda y = \left(\frac{d}{dx} - \lambda\right) = f(x) \qquad (\lambda \in \mathbb{C}),$$

With complex forcing term $f : I \subset \mathbb{R} \to \mathbb{C}$ continuous in $I \ni 0$,has the general solution

$$y(x) = e^{\lambda x}\int e^{-\lambda x} f(x)dx$$

$$= e^{\lambda x}\int_o^x e^{-\lambda t} f(t)dt + ke^{\lambda x}$$

$$= \int_0^x g_\lambda(x - t)f(t)dt + kg_\lambda(x)$$

$$(x \in I, k = y(0) \in \mathbb{C}) \ldots..(9)$$

Here $g_\lambda(x) = e^{\lambda x}$is the impulsive response of the differential operator $\left(\frac{d}{dx} - \lambda\right)$. It is the (unique) solution of $y' - \lambda y = 0, y(0) = 1$. Formula (9) can be proved as (3) in the real case. In particular, the solution of the first-order problem

$$\begin{cases} y' - \lambda y = f(x), \\ y(0) = 0 \end{cases}$$

$(\lambda \in \mathbb{C})$ is unique and is given by

$$y(x) = \int_0^x e^{\lambda(x-t)} f(t)dt.$$

…….. (10)

The following result gives a particular solution of (5) as a convolution integral.

**Theorem :1** Let $g \in c^0(I), 0 \in I$, and let $y_0, y_0'$ be two arbitrary real number. Then the initial value problem

$$\begin{cases} y'' + ay' + by = g(x), \\ y(0) = y_{0,} \quad y'(0) = y_0' \end{cases}$$

…….. (11)

Has a unique solution, defined on the whole of $I$, and given by

$$y(x) = \int_0^x h(x-t)g(t) + (y_0' + ay_0)h(x) + y_0 h'(x) \quad (x \in I).$$

……(12)

In particular (taking $f = 0$), the solution of the homogenous problem

$$\begin{cases} y'' + ay' + by = 0, \\ y(0) = y_0, \; y'(0) = y_0' \end{cases}$$

Is unique, of class $C^\infty$ on the whole of $\mathbb{R}$, and is given by

$$y_h(x) = (y_o' + ay_0)g(x) + y_0 g'(x) \quad (x \in \mathbb{R}).$$

…..(13)

**Proof:**

The uniquess of the solutions of the problem (11) follows from the fact that if $y_1$ and $y_2$ both solve (11), then their difference $\overline{y} = y_1 - y_2$ solves the problem whence $y = 0$ by Theorem 1. Now note that the function $g'$ satisfies the homogenous equation. Indeed, since $L$ has constant coefficients, we have

$$Lg' = L\frac{d}{dx}g$$

$$= \left[(\tfrac{d}{dx})^2 + a\tfrac{d}{dx} + b\right]\tfrac{d}{dx}g$$

$$= \frac{d}{dx}Lg$$

$$= 0$$

By the linearity of $L$ and by Theorem 1 it follows that the function $y$ given by (12) satisfies $(Ly)(x) = f(x) = f(x)\forall \in I$. It is immediate that $y(0)$ =Finally, since

$$y'(x) = \int_0^x g'(x-t)f(t)dt + (y_0' + ay_o)g'(x) + y_0 g''(x),$$

We have

$$y'(x) = y_0' + ay_0 + y_0 g''(0)$$

$$= y_0' + (ay_0 + y_o(-ag'(0) - bg(0))$$

$$= y_0'$$

It is also possible to give a constructive proof, analogous to that of theorem1. Indeed, by proceeding as in the proof of this theorem and using ( 9 ), we find that $y$ solves the problem (11) if and only if $y$is given by

$$y(x) = \int_0^x g(x-s)f(s)ds + (y_0' - \lambda_2 y_0 g(x) + y_0 e^{\lambda_2 x}.$$

This formula agrees with (12) in view of the equality

$$e^{\lambda_2 x} = g'(x) - \lambda_1 g(x),$$

which follows from

$$g(x) = \int_0^x e^{\lambda_2(x-t)} e^{\lambda_1 t} dt \quad x \epsilon \mathbb{R}$$

…...(14)

by interchanging $\lambda_1$ and $\lambda_2$ (to see that formula ( 14 ) is symmetric in $\lambda_1 \leftrightarrow \lambda_2$,just make the change of variables $x - t = s$ in the integral with respect to $t$).

Hence proved.

**Example:** Solve the initial value problem

$$\begin{cases} y'' + y = \frac{1}{cosx}, \\ y(0) = 0, y'(0) = 0, \end{cases}$$

**Solution:**

We have that $p(\lambda) = \lambda^2 + 1,$

Whence $\lambda_1 = \overline{\boldsymbol{\lambda_2}} = \boldsymbol{i}$ and the impulsive response is

$$g(x) = \sin x.$$

The initial data are given at $x = 0$ and we can work in the interval $I = (-\pi/2, \pi/2),$

Where the forcing term $f(x) = \frac{1}{cosx}$ is continuous.

Using equ (12),we get

$$y(x) = \int_0^x \sin(x-t)\frac{1}{cost}dt$$

$$= \int_0^x (sinxcost - cosxsint)\frac{1}{cost}dt$$

$$= sinx\int_0^x dt - cosx\int_0^x \frac{sint}{cost}dt$$

$$= \boldsymbol{xsinx + cosxlog(cosx)}$$

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