EXISTENCE OF SOLUTIONS TO INITIAL-VALUE PROBLEM FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract

We consider existence of solutions to initial-value problems for second-order singular differential equations. We observe that the existence can be demonstrated in terms of simple initial-value problem. Local existence and uniqueness of solutions are proven. Under the conditions which are weaker than previously known conditions.

Keywords: Initial - value problem, singular differential equation, emden -fowler equation.

Introduction

In this paper, we study the singular initial value problems (IVPs) of the type

$$y'' + 2t^{-1}y' + y^n(t) = 0,$$

$$y(0) = 1, y'(0) = 0, \qquad (1)$$

have seeked the concentration of many mathematicians and physicists. Our aim of this paper to study the more general IVPs of the form

$$y'' + p(t)\, y' + q\big(t, y(t)\big) = 0,$$

$$y(0) = \mathrm{a}, y'(0) = \mathrm{b}, t > 0 \qquad (2)$$

and to make further progress beyond the achievements made so far in this regard. The case $q = f(t)g(x)$ corresponds to Emden - Fower equations[10]. In above equation (2), the function $p(t)$ may be singular at t = 0.It prolong some well–known IVPs in the literature[1,7]

In the case b = 0 the existence of the solution for the problem (2) has been studied in [2],where the authors illustrated the importance of the condition b=0 for the existence. We find the conditions for $p(t)$ and $q(t, y(t))$ to guarantee the existence of the solution for $b \neq 0$.

Existence Theorems

We say that $y(t)$isasolution to (2) if and only if there exists some $T > 0$ such that

(1) $y(t)$ and $y'(t)$ are absolutely continuous on $[0, T]$,

(2) $y(t)$ satisfies the equation given in (2) a.e. on $[0, T]$,

(3) $y(t)$ satisfies the initial condition given in (2).

And we generalize the existence theorem of solutions in [2].

Theorem 1. Let p and q satisfy the following conditions:

(1) p is measurable on [0, 1];

(2) $p \geq 0$;

(3) $\int_0^1 sp(s)\, ds < \infty$;

(4) there exist α , β with $\alpha < a < \beta$ and $K > 0$ such that

(a) for each $t \in (0,1]$, $q(t,.)$ is continuous on $[\alpha ,\beta]$;

(b) for each $y \in [\alpha ,\beta]$, $q(., y)$ is measurable on [0 ,1];

(c) $|q(t, y)| \leq K$.

Then a solution to the initial – value problem (2) with b = 0 exists.

In [4] the author illustrated the importance of the condition b = 0 for the existence.

To overcome the difficulties in the case $b \neq 0$ we consider a generalization of theorem 1 and show that the statement of the theorem is true without condition (c) and with weaker conditions on $q(t, y)$.

Theorem 2. Suppose that $p(t)$ is integrable on the interval $[c, d]$ for all $c > 0$ and p and q satisfy the following conditions:

(1) p is measurable on [0, 1];

(2) $p \geq 0$;

(3) there exist α , β with $\alpha < a < \beta$ and $K > 0$, and an integrable (improper, in general) $\varphi(t)$ such that

(a) for each $t \in (0,1]$, $q(t,.)$ is continuous on $[\alpha ,\beta]$;

(b) for each $x \in [\alpha, \beta]$, $q(., x)$ is measurable on [0 ,1];

(c) $|q(t,y) - \varphi(t)| \leq K$.

Then a solution to the initial – value problem (2) exists for all b ∈ R such that

$$b = z'(0), \quad (3)$$

where $z(t) \in C[0,1]$ is a solution of the problem

$$z'' + p(t)z' + \varphi(t) = 0,$$

$$z(0) = a,\ z'(0) = b,\ t > 0. \quad (4)$$

That is, the existence of the problem (4) for some $\varphi(t)$. For the problems with b = 0, the initial-value problem (4) always has a solution $z(t) = a$, for $\varphi(t) = 0$. So Theorem 1 corresponds to the cases $\varphi(t) = 0$ and $z(t) = a$.

The advantages of Theorem 2 is that the problem (4) always has a solution for some appropriate $\varphi(t)$; for example, for $\varphi(t) = -bp(t)$, it has a solution $z(t) = a + bt$. The conclusion of the theorem remains valid for all solutions of (4).

It is also clear from the conclusion of Theorem 2 that the interval [0,1] can be taken as $[0, t_0]$ for some small enough $t_0 > 0$.

Proof :

For $t \in (0,1]$, we define the functions

$$h(t) = \exp\left(\int_1^t p(s)ds\right) \geq 0,$$

$$h_1(t) = \exp\left(-\int_1^t p(s)ds\right), \quad (5)$$

$$E(t) = \int_1^t h_1(t)\, ds.$$

where $h(t)$ is a bounded function and contionuous for $t \in (0,1]$. It is continuous or has a removable discontinuity at $t = 0$ and is differentiable a.e.

Show that the problem (2) is equivalent to the following integral equation

$$y(t) = \int_0^t E(s)\, e^{\int_1^s p(\tau)d\tau} - E(t)e^{\int_1^s p(\tau)d\tau} \times [q(s,y(s)) - \varphi(s)]ds + z(t). \quad (6)$$

Let us show the existence of the integral in (6). For any $\delta > 0$,we have

$$\left|\int_\delta^t E(s)\, e^{\int_1^s p(\tau)d\tau}[q(s,y(s)) - \varphi(s)]ds\right|$$

$$\leq k\left|\int_\delta^t E(s)\, e^{\int_1^s p(\tau)d\tau} ds\right| \quad (7)$$

$$= k\left|\int_\delta^t \int_1^s h_1(u)\, e^{\int_1^s p(\tau)d\tau} duds\right|$$

$$= \left|\int_\delta^t \int_1^s e^{-\int_1^u p(v)dv}\, e^{\int_1^s p(\tau)d\tau} duds\right|.$$

It follows from $u \geq s$ on the set $[s, 1] \times [0,t]$ that

$$e^{-\int_1^u p(v)dv} e^{\int_1^s p(\tau)d\tau} = e^{-\int_1^u p(v)dv} \leq 1 \quad (8)$$

$$\left|\int_\delta^t E(s)\, e^{\int_1^s p(\tau)d\tau}[q(s,y(s)) - \varphi(s)]ds\right|$$

$$\leq k\left|t - \frac{t^2}{2}\right| \quad (9)$$

Likewise, we obtain

$$\left|\int_\delta^t E(t)\, e^{\int_1^s p(\tau)d\tau}[q(s,y(s)) - \varphi(s)]ds\right|$$

$$\leq k\left|\int_\delta^t E(t)\, e^{\int_1^s p(\tau)d\tau} ds\right|$$

$$= k\left|\int_\delta^t \int_1^t h_1(u) e^{\int_1^s p(\tau)d\tau} duds\right|$$

$$\leq k|t - t^2| \quad (10)$$

So the right-hand side of (6) makes sense for any $p(t) \geq 0$ and $|[q(s,y(s)) - \varphi(s)]| \leq k$ and

$$\lim_{\delta\to 0} \int_\delta^t (E(s)\, e^{\int_1^s p(\tau)d\tau} - E(t)e^{\int_1^s p(\tau)d\tau}) \times [q(s,y(s))\varphi(s)]\, ds + z(t) = \int_0^t (E(s)\, e^{\int_1^s p(\tau)d\tau} - E(t)e^{\int_1^s p(\tau)d\tau} \times [q(s,y(s))\varphi(s)]ds + z(t). \quad (11)$$

Now calculate the derivatives $y'(t)$ and $y''(t)$ from (6) by using the Leibniz rule:

$$y'(t) = \Big(\int_0^t E(s)\, e^{\int_1^s p(\tau)d\tau}[q(s,y(s)) - \varphi(s)]ds - \int_0^t E(t)\, e^{\int_1^s p(\tau)d\tau}[q(s,y(s)) - \varphi(s)]ds + z(t)\Big)'$$

$$= E(t)e^{\int_1^s p(\tau)d\tau}[q(t,y(t))\varphi(t)] - E'(t)\int_0^t e^{\int_1^s p(\tau)d\tau}[q(s,y(s))\varphi(s)]ds - E(t)e^{\int_1^t p(\tau)d\tau}[q(t,y(t)) - \varphi(t)] + z'(t)$$

$$= -h_1(t)\int_0^t e^{\int_1^s p(\tau)d\tau}[q(s,y(s)) - \varphi(s)]ds + z'(t),$$

$$y''(t) = \Big(-h_1 t)\int_0^t e^{\int_1^s p(\tau)d\tau}[q(s,y(s)) - \varphi(s)]ds + z'(t)\Big)'$$

$$= -h_1'(t)\int_0^t e^{\int_1^s p(\tau)d\tau}[q(s,y(s)) - \varphi(s)]ds - h_1(t)\, e^{\int_1^t p(\tau)d\tau}[q(t,y(t)) - \varphi(t)] + z''(t)$$

$$= -h_1'(t)\int_0^t e^{\int_1^s p(\tau)d\tau}[q(s,y(s)) - \varphi(s)]ds - [q(t,y(t)) - \varphi(t)] + z''(t). \quad (12)$$

It follows from (12) that

$$x''(t) + p(t)x'(t) + q(t,y(t))$$

$$= -h_1'(t)\int_0^t e^{\int_1^s p(\tau)d\tau}[q(s,y(s))$$

$$-\varphi(s)]ds - [q(t,y(t)) - \varphi(t)] + z''(t)$$

$$-p(t)\, h_1(t)\int_0^t e^{\int_1^s p(\tau)d\tau}[q(s,y(s)) -$$

$\varphi(s)]ds + p(t)z'(t) + q(t,y(t))$ (13)

$= z''(t) + p(t)z'(t) + \varphi(t)$

$= 0.$

That is, the problem (2) is equivalent to (4). We define the recurrence relations

$$y_0(t) = z(t), \quad (14)$$

In general,

$$y_n(t) = \int_0^t (E(s)\, e^{\int_1^s p(\tau)d\tau} - E(t)\, e^{\int_1^s p(\tau)d\tau})$$

$$\times[q(s, x_{n-1}(t)) - \varphi(s)]ds + z(t),$$

(15)

where $z(t)$ is a solution of the problem (4). It follows from (9), (10), and (14) that $\alpha < y_n(t) < \beta$ for $\alpha < y_{n-1}(t) < \beta$ and for small enough $t \in [0,t_0)$.

For $t_1, t_2 \in [0, t_0)$, from equation (9) and (10),we have

$|y_n(t_2) - y_n(t_1)| =$

$$\left| \int_{t_1}^{t_2} (E(s)\, e^{\int_1^s p(\tau)d\tau} - E(t)\, e^{\int_1^s p(\tau)d\tau})\right.$$

$$\left. \times [q(s, y_{n-1}(s)) - \varphi(s)]ds \right|$$

$$\le 2K[\, (t_2 - \tfrac{t_2^2}{2}) - (t_1 - \tfrac{t_1^2}{2})]$$

$$\le 2K(t_2 - t_1)(1 + \tfrac{t_1}{2} + \tfrac{t_2}{2})$$

$$\le K(t_2 - t_1). \quad (16)$$

for some constant K_1. Thus, the sequence $y_n(t)$ is uniformly bounded and uniformly continuous. By using Ascoli – Arzela lemma, there exists a continuous $y(t)$ such that $y_{n_k}(t) \to y(t)$ uniformly on [0,T], for any fixed T∈ $[0, t_0)$. Without loss of generality, say $y_n(t) \to y(t)$. Then

$$y(t) = \lim_{n\to\infty} \int_0^t (\, E(s)\, e^{\int_1^s p(\tau)d\tau} - E(t)\, e^{\int_1^s p(\tau)d\tau})$$

$$\times[q(s, y_n(s)) - \varphi(s)]ds + z(t) \quad (17)$$

$$\int_0^t (E(s)\, e^{\int_1^s p(\tau)d\tau} - E(t)\, e^{\int_1^s p(\tau)d\tau})$$

$$\times [q(s, y(s)) - \varphi(s)]ds + z(t),$$

using the lebesgue dominated convergence theorem.

Note that the positivity condition of the function $p(t)$ can be weakened. the positivity of $p(t)$ has been used in the proof of Theorem 2 to show the (removable) continuity of the function $h(t)$ at 0. Assuming that the following condition holds

(i) $|p|$ is integrable on $[c, d]$

for any fixed $c,d \in (0, 1]$, $c < d$ and

$$M \le \int_c^d p(s)ds < +\infty;$$

for some fixed M (18)

we can prove a similar theorem

Theorem 3. The conclusion of the Theorem 2 remains valid if condition (2) is replaced by (i) .

Proof: We need to make some modifications to the proof of Theorem 2; for example, instead of the inequality

$$e^{-\int_1^u p(v)dv} e^{\int_1^s p(\tau)d\tau} \le 1, \quad (19)$$

for $u \ge s$, we have

$$e^{-\int_1^u p(v)dv} e^{\int_1^s p(\tau)d\tau} = e^{-\int_s^u p(v)dv} \le e^{-L},$$

(20)

for small enough u and s. Note that the existence of the solution of the problems like

$$y'' + (\tfrac{a_m}{t^m} + \tfrac{a_{m-1}}{t^{m-1}} + \ldots + \tfrac{a_1}{t} + A(t)y' + q(t, y(t)) = 0,$$

$$x(0) = a,\ x'(0) = b, t > 0, \quad (21)$$

follows from theorem 2, where $A(t)$ is differentiable function, $q(t,x)$ satisfies the conditions (3), a_1, $a_2, \ldots, a_m$ are real constants, and $a_m > 0$. Indeed for small enough t we have $p(t) > 0$ and therefore the hypotheses of theorem 2 and 3 are true for small enough $t \in [0,T]$; for $b = 0$ the problem (4) has a solution $z(t) = a$, and so (21) has a solution for all bounded $q(t, y(t))$ with carathedory conditions, but for $b \ne 0$ the problem (21) has a solution for $q(t, y(t))$ with

$$|\, q(t, y(t)) + b(\tfrac{a_m}{t^m} + \tfrac{a_{m-1}}{t^{m-1}} + \ldots + \tfrac{a_1}{t})| < K$$

some small enough neighbourhood of 0, since the corresponding problem (4) can be taken (e.g.) as

$$z'' + (\tfrac{a_m}{t^m} + \tfrac{a_{m-1}}{t^{m-1}} + \ldots + \tfrac{a_1}{t} + A(t))z' - b(\tfrac{a_m}{t^m} + \tfrac{a_{m-1}}{t^{m-1}} + \ldots + \tfrac{a_1}{t} + A(t)) = 0,$$

$$z(0) = a, z'(0) = b, t > 0, \quad (22)$$

has a solution $z(t) = a + bt$. For $b \ne 0$ the condition $q(t, y(t))$ can be changed by using different functions for $\varphi(t)$ can be taken as

$$\varphi(t) = \tfrac{b_m}{t^m} + \tfrac{b_{m-1}}{t^{m-1}} + \ldots$$

$$= -\tfrac{b a_m}{t^m} + \tfrac{1}{t^{m-2}}(\tfrac{b a_{m-1}}{a_m} - b a_{m-2})$$

$$+ \tfrac{1}{t^{m-3}}(\tfrac{b a_{m-1} a_{m-2}}{a_m} - b a_{m-3})$$

$$+ \tfrac{1}{t}(\tfrac{b a_{m-1} a_2}{a_m} - b a_1) + \tfrac{b a_{m-1} a_1}{a_m} - bA(t) - \tfrac{b a_{m-1}}{a_m}$$

$$z'' + (\tfrac{a_m}{t^m} + \tfrac{a_{m-1}}{t^{m-1}} + \ldots + \tfrac{a_1}{t} + A(t))z' + \varphi(t) = 0,$$

$z(0) = a, z'(0) = b, t > 0,$ (24)

with solution $z(t) = a + bt - \left(\frac{ba_{m-1}}{2a_m}\right) t^2.$

continuing this process, the condition $q(t, y(t))$ can be reduced to $| q(t, y(t)) + \frac{ba_m}{t^m} | < K.$

The inequalities (7),(8),(9) and (10) can be easily established for the function $q(t, y)$ with

$|q(t, y(t)) - \varphi(t)| \le m(t),$ (25)

where $m(t)$ is absolutely integrable function.

Applications

By using existence and uniqueness criteria, we can find the wide classes of the initial- value problems. Adding a function $\varphi(t)$ to $q(t, y)$ in the class of solvable problem, it can be extended, where $\varphi(t)$ is taken from equation (4) with a solution.

$y'' + p(t)y' + \varphi(t, y(t)) = 0,$

$y(0) = a, y'(0) = b, t > 0,$ (26)

has a solution,then

$y'' + p(t)y' + \varphi(t, y(t)) + q(t, y(t)) = 0,$

$y(0) = a, y'(0) = b, t > 0,$ (27)

where $q(t, y)$ is a bounded function with caratheodary conditions, has also a solution.

Example. The problem

$y'' + p(t)y' + q(t, y(t)) - bp(t) = 0,$

$y(0) = a, y'(0) = b, t \ge 0,$ (28)

has a solution for all bounded $q(t, y(t))$. Indeed the problem

$z''(t) + p(t)z'(t) - bp(t) = 0,$

$z(0) = a, z'(0) = b,$ (29)

has a solution $z(t) = bt + a$ Then the existence of solution of (28) follows from Theorem 2.

Conclusion

We extended the class of second order- singular IVPs and established difficulties related to the singularity overcome for the problem (2) with $p \ge 0$ or

$M \le \int_c^d p(s)ds < +\infty;$

for some fixed M. (30)

The existence of a solution reduced to finding a solution some problems like (4). The conditions are weaker than the previously known is obtained and can be easily reduced to several special cases.

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