COVER-INCOMPARABILITY GRAPHS AND DELTA- PRESERVING 3-COLORED DIAGRAMS OF POSETS

Baiju Sukumaran
Department of Mathematics, Sree Narayana College, Punalur
India, PIN – 691305

Abstract : The cover-incomparability graph of a poset P is the edge-union of the covering and the incomparability graph of P. Here we use 3-colored diagrams to characterize the forbidden ⊲ - preserving subposets of the posets whose cover-incomparability graphs are not line graphs is proved.

***IndexTerms* -** Cover-incomparability graph, Linegraph, Poset.

1 Introduction and preliminaries

Cover-incomparability graphs of posets, or shortly C-I graphs, were introduced in [2] as the underlying graphs of the standard interval function or transit function on posets (for more on transit functions in discrete structures cf. [3, 4, 5, 6, 11]). On the other hand, C-I graphs can be defined as the edge-union of the covering and incomparability graph of a poset; in fact, they present the only non-trivial way to obtain an associated graph as unions and/or intersections of the edge sets of the three standard associated graphs (i.e. covering, comparability and incomparability graph). In the paper that followed [9], it was shown that the complexity of recognizing whether a given graph is the C-I graph of some poset is in general NP-complete. In [1] the problem was investigated for the classes of split graphs and block graphs, and the C-I graphs within these two classes of graphs were characterized. This resulted in a linear-time recognition algorithms for C-I block and C-I split graphs. It was also shown in [1] that whenever a C-I graph is a chordal graph, it is necessarily an interval graph, however a structural characterization of C-I interval graphs (and thus C-I chordal graphs) is still open. Recently C-I distance-hereditary graphs have been characterized and shown to be efficiently recognizable [10].

Let P = (V;_≤) be a poset. If u ≤ v but u ≠ v, then we write u < v. For u, v ϵ V we say that v *covers* u in P if u < v and there is no w in V with u < w < v. If u ≤ v we will sometimes say that u is *below* v, and that v is *above* u. Also, we will write u ⊲ v if v covers u; and u ⊲⊲ v if u is below v but not covered by v. By u ‖ v we denote that u and v are incomparable. Let V' be a nonempty subset of V . Then there is a natural poset Q = (V';≤ $'$), where u ≤ $'$ v if and only if u ≤ v for any u,v ϵ V'. The poset Q is called a *subposet* of P and its notation is simplified to Q = (V';≤). If, in addition, together with any two comparable elements u and v of Q, a chain of shortest length between u and v of P is also in Q, we say that Q is an isometric subposet of P. Recall that a poset P is *dual* to a poset Q if for any x, y ϵ P the following holds: x ≤ y in P if and only if y ≤ x in Q. Given a poset P, its cover-incomparability graph G_P has V as its vertex set, and uv is an edge of G_P if u ⊲ v, v ⊲ u, or u and v are incomparable. A graph that is a cover-incomparability graph of some poset P will be called a C-I graph.

Lemma 1 [2] Let P be a poset and G_P its C-I graph. Then

(i) G_P is connected;
(ii) vertices in an independent set of G_P lie on a common chain of P;
(iii) an antichain of P corresponds to a complete subgraph in G_P ;
(iv) G_P contains no induced cycles of length greater than 4.

2. 3-colored diagrams

A 3-coloured diagram Q; we consider normal edges to represent vertices in a covering relation and red edges to represent incomparable vertices or vertices in a covering relation and dashed lines to represent a chain of length three and thus constitute the 3-colors and hence the name 3-*colored* diagram. The idea of 3-colored diagrams is explained as follows. Let G be a C-I graph and H be an induced subgraph of G. We note that there can be different ⊲- preserving subposets Q_i of some posets with G_{Q_i} isomorphic to the subgraph H. Let u, v,w be an induced path in the direction from u to v in H. There are four possibilities in which u, v and w can be related in the ⊲ - preserving subposets. It is possible to have u ⊲ v, u ‖ v, v ⊲ w and v ‖ w. Each case will appear as a ⊲ - preserving subposet of four different posets. If u ⊲ v and v ⊲ w in a subposet, then u ⊲ v ⊲ w is a chain in the subposet and u, v,w is an induced path in H. If there is either u ‖ v or v ‖ w in a subposet Q, then there should be another chain from u to w in Q in order to have u, v,w an induced path in H. We try to capture this situation using the idea of 3-colored diagram. Suppose in ⊲ - preserving subposet Q of a poset P, there exists two elements u, v which is always connected by some chain of length three in Q. Let w be an element in Q such that either both uw and vw are red edges or any one of them is a red edge. Then in order to have a chain between u and v, there must exist an element x in Q so that u, x, v form a chain in Q. When both edges are normal, then we have the chain u,w, v in Q and hence the chain u, x, v is not required in this case. We denote the chain u, x, v by dashed lines between ux and xv in order to specify that it is possible to have the presence or absence of the chain u, x, v in Q. The presence of the chain u, x, v implies that either both of the edges uw and wv are red edges or one of them is a red edge. The absence of the chain implies that both uw and vw are normal edges in Q. We call posets having the above mentioned diagrams as 3-colored diagrams. Thus a 3-colored diagram contains normal edges, red edges and dashed lines, in which the dashed line between elements u and v will vanish, when there is a chain between u and v using normal or red edges. We can define 3-colored subposets in a similar way as discussed above. All subposets of the poset P that we consider in this paper are 3-colored diagrams. Thus by a single 3-colored diagram, we represent a collection of ⊲ - preserving subposets to be forbidden for a poset. We sometimes use the term 3-colored subposets instead of 3-colored diagrams in this paper. In a similar way the dual of a 3-colored diagram is also meaningful and represents a collection of ⊲- preserving dual subposets.

Theorem 2 (Theorem 1,[8]): Let $\mathscr{G}$ be a class of graphs with a forbidden induced subgraphs characterization. Let $\mathscr{P}$= {P | P is a poset with $G_{T_P} \in \mathscr{G}$ }. Then $\mathscr{P}$ has a characterization by forbidden ⊲ - preserving subposets.

Theorem 3 (Theorem 7.1.8, [7]) Let G be a graph. Then G is a line graph if and only if G contains none of the nine forbidden graphs of Figure 1 as an induced subgraph.

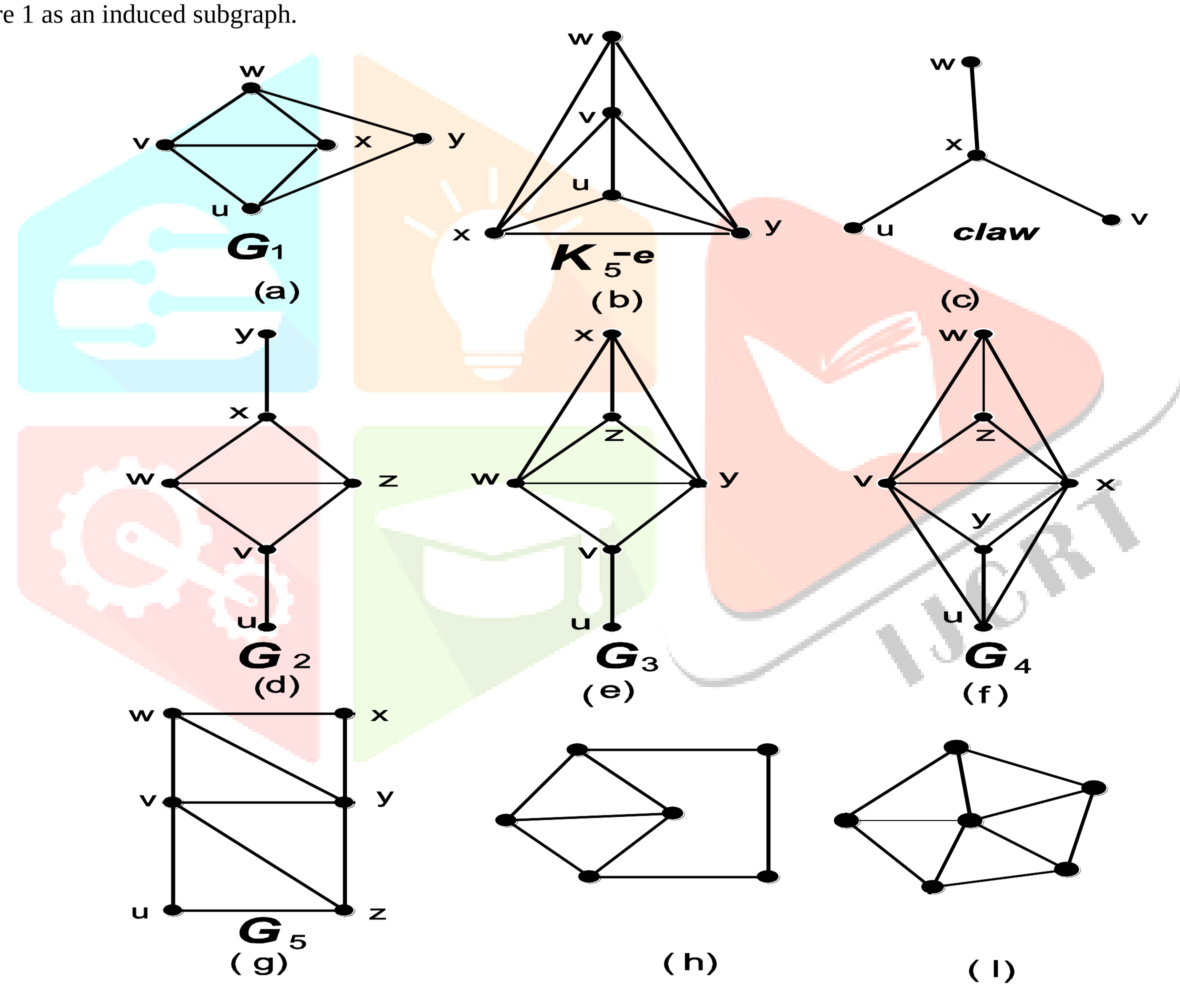

Figure 1: Nine Forbidden induced subgraphs of line graphs

We consider the 3-colored subposets to be forbidden so that its C-I graphs belong to the graph family $\mathcal{F}(G_2)$ of G_2 in Figure1

3. 3-colored ⊲ - preserving subposets of posets whose C-I graphs belong to the family $\mathcal{F}(G_2)$

We have the following theorem regarding the graph family $\mathcal{F}(G_2)$.

Theorem 4 If P is a poset, then G_P belongs to $\mathcal{F}(G_2)$ if and only if P contains the 3-colored diagram Q_1 from Figure 3 and their duals.

Proof. Suppose P contains the 3-colored diagram Q_1. Then since w and z are incomparable in P, the set of vertices { u, v, w, x, y, z} induce the graph G_2 from Figure 1(d).

Conversely, suppose $G_P \epsilon \mathcal{F}(G_2)$. Then G_P contains an induced subgraph G_2 shown in Figure 1(d), with vertices labeled by u, v, w, x, y and z. The set of vertices **{u,** w, y} is an independent set in G_2. Therefore these vertices lie on a common chain in P (by Lemma 1*(ii)*) and they are not in a covering relation. Denote the chain containing {u,w, y} by C. Then the following cases (i) and (ii) cannot occur.

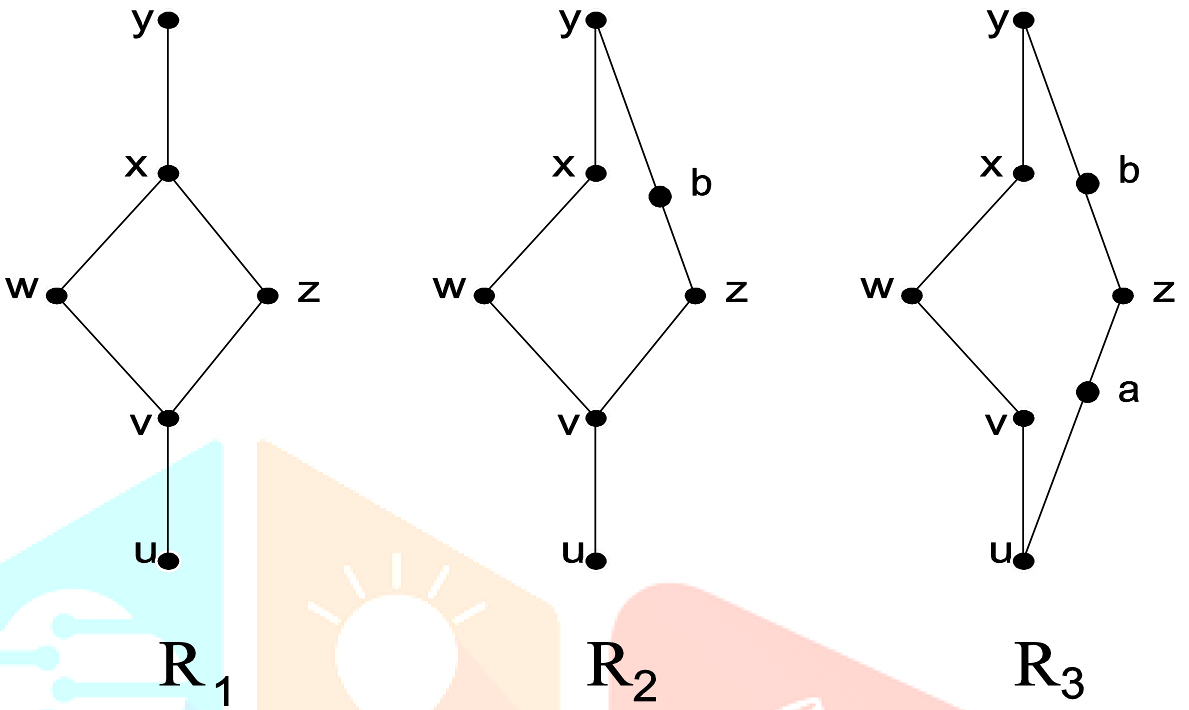

Figure 2: $\lhd$ - preserving subposets corresponding to Q_1

(i): u $\lhd\lhd$ y $\lhd\lhd$ w

Since v and y are nonadjacent in G they lie on a common chain in P. v $\lhd\lhd$ y: then v $\lhd\lhd$ y $\lhd\lhd$ w in P, contradicting v and w are adjacent in G. y $\lhd\lhd$ v: then u $\lhd\lhd$ y $\lhd\lhd$ v in P, contradicting u and v are adjacent in G. The same contradiction arise if w $\lhd\lhd$ y $\lhd\lhd$ u.

(ii): w $\lhd\lhd$ u $\lhd\lhd$ y:

Since u and x are nonadjacent in G, they lie on a common chain in P. u $\lhd\lhd$ x: then w $\lhd\lhd$ u $\lhd\lhd$ x in P, contradicting w and x are adjacent in G. x $\lhd\lhd$ u: then x $\lhd\lhd$ u $\lhd\lhd$ y in P, contradicting x and y are adjacent in G. The same contradiction arise if y $\lhd\lhd$ u $\lhd\lhd$ w.

The only possible cases are u $\lhd\lhd$ w$\lhd\lhd$ y and y $\lhd\lhd$ w$\lhd\lhd$ u. Without loss of generality, assume that u $\lhd\lhd$ w $\lhd\lhd$ y. Since v is adjacent to u and w and x is adjacent to w and y in G, we have u $\lhd$ v $\lhd$ w $\lhd$ x $\lhd$ y: then we have two possibilities.

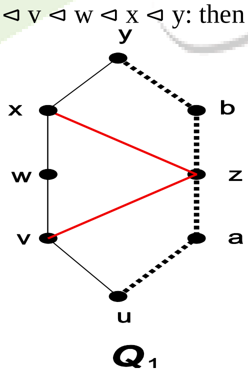

Figure 3: Forbidden 3-colored diagrams for posets whose C-I graphs contains G_2, depicted in Figure 1 (d).

Case (1): v $\lhd$ z in P: again we have two possibilities.

Subcase (1.1): z $\lhd$ x in P: take (v, z) and (x, z) as normal edges and avoid all dashed lines in Figure 3 to obtain the $\lhd$ - preserving subposet R_1 in Figure 2.

Subcase (1.2): z || x in P: take the chain from y to z through b to obtain the $\lhd$- preserving subposet R_2 in Figure 2.

Case (2): v || z in P: take the chain from u to z through a to obtain the $\lhd$- preserving subposet R_3 in Figure 2. All posets in Figure 2 are represented by a single 3-colored diagram Q_1, see Figure 3. □

Remarks

The number of forbidden $\lhd$ - preserving subposets of a poset P is such that its C-I graph G_P belongs to a graph possessing a forbidden induced subgraph characterization as instances of the Theorem 2 is in general very large compared to the number of forbidden induced subgraphs. The idea of 3-colored diagrams is introduced to shorten the list of forbidden $\lhd$ - preserving subposets.

References

[1] B. Brešar, M. Changat, T. Gologranc, J. Mathew, A. Mathews, Coverincomparability graphs and chordal graphs, Discrete Appl. Math. 158 (2010) 1752-1759.

[2] B. Brešar, M. Changat, S. Klavžar, M. Kovše, J. Mathew, A. Mathews, Coverincomparability graphs of posets, Order 25 (2008) 335-347.

[3] B. Brešar, M. Changat, S. Klavžar, J. Mathew, A. Mathews, Prasanth G. N., Characterizing posets for which their natural transit functions coincide, Ars Math. Contemp. 2 (2009) 27-33.

[4] M. Changat, S. Klavžar and H. M. Mulder, The all-paths transit function of a graph, Czech. Math. J. 51(126) (2001) 439-448.

[5] M. Changat and J. Mathews, Induced path transit function, monotone and Peano axioms, Discrete Math. 286 (2004) 185-194.

[6] A. Mathews and J. Mathews, Transit functions on posets and lattices. In: Convexity in Discrete Structures (M. Changat, S. Klavžar, H.M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 105-116.

[7] A. Brandstädt, V. B. Le, J. P. Spinrad: Graph Classes: A Survey, SIAM Monographs on Discrete Mathematics and Applications, 3, 1999.

[8] J.Book , J. Maxová, Characterizing subclasses of cover-incomparability graphs by forbidden subposets,Order manuscript.

[9] J. Maxová, P. Pavlìková, D. Turzìk, On the complexity of cover-incomparability graphs of posets, Order 26 (2009), 229-236.

[10] J. Maxová, D. Turzìk, Which distance-hereditary graphs are cover-incomparability graphs?, Discrete Appl. Math.161(2013), 2095-2100.

[11] H. M. Mulder, Transit functions on graphs (and posets). In: Convexity in Discrete Structures (M. Changat, S. Klav_zar, H.M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 117-130.