A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY FRACTIONAL CALCULUS WITH FIXED POINTS

[1]Dr. Balvir Singh, [2]Dr. Saurabh Porwal

[1] Assoc. Professor, [2] Lecturer

[1] Department of Mathematics

R P Degree College, Kamalganj, Farrukhabad-209601

(U.P.) India

e-mail: balvirsingh.rp@gmail.com,

[2] Sri Radhey Lal Arya Inter College, Aihan, Hathras

(U.P.) India

e-mail: saurabhjcb@rediffmail.com

Abstract: In this paper, authors introduce a new subclass $S^*_{H,\lambda}(\alpha,t,z_0)$ of S_H by using fractional calculus. We give univalence criteria and sufficient coefficient condition for normalized harmonic functions belonging to the class $S^*_{H,\lambda}(\alpha,t,z_0)$ where $0\le\alpha<1, 0\le\lambda<1$ and $0\le t\le 1$. These coefficient condition are also shown to be necessary for subclass $TS^*_{H,\lambda}(\alpha,t,z_0)$ of $S^*_{H,\lambda}(\alpha,t,z_0)$ in which *h* has negative and *g* has positive coefficients. This leads to extreme points, distortion bounds and radius of convexity. We also discuss a class preserving integral operator and show that the class studied in this paper is closed under convolution and convex combinations.

Keywords. Harmonic, univalent, starlike, Owa-Srivastava fractional calculus operator.

AMS 2010 Mathematics Subject Classification: 30C45, 30C50.

I. INTRODUCTION

A continuous complex-valued function $f=u+iv$ defined in a simply connected complex domain *D* is said to be harmonic in *D* if both *u* and *v* are real harmonic in *D*. In any simply connected domain we can write $f=h+\bar{g}$, where *h* and *g* are analytic in *D*. We call *h* the analytic part and *g* the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in *D* is that $|h'(z)|>|g'(z)|, z\in D$. See Clunie and Sheil-Small [2].

Denote by S_H the class of functions $f=h+\bar{g}$ which are harmonic univalent and sense-preserving in the unit disk $U=\{z:|z|<1\}$ for which $f(0)=f_z(0)-1=0$. Then for $f=h+\bar{g}\in S_H$ we may express the analytic functions *h* and *g* as

$$h(z)=z+\sum_{k=2}^{\infty}a_k z^k,\quad g(z)=\sum_{k=1}^{\infty}b_k z^k,\quad |b_1|<1. \tag{1}$$

Recently, Jahangiri [6] defined the class $TS^*_H(\alpha)$ consisting of function $f=h+\bar{g}$ such that *h* and *g* are of the form

$$h(z)=z-\sum_{k=2}^{\infty}|a_k|z^k,\quad g(z)=\sum_{k=1}^{\infty}|b_k|z^k, \tag{2}$$

Which satisfy the condition

$$\Re\left\{\frac{zh'(z)-\overline{zg'(z)}}{h(z)+g(z)}\right\}\geq\alpha, \qquad (0\leq\alpha<1)$$

The case when $\alpha=0$ is given in [12] and for $\alpha=b_1=0$ see [11].

The class S_H reduces to class S of normalized analytic univalent functions if co-analytic part of f i.e. $g\equiv 0$. For this class $f(z)$ may be expressed as

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k. \tag{3}$$

Several authors such ([3], [4], [5], [9], [11] [12]), studied the sub-classes of analytic univalent functions by using fractional calculus operator. In this paper, an attempt has been made to study the subclass of harmonic univalent functions by using fractional calculus with fixed points in the following way

$$h(z)=a_1 z-\sum_{k=2}^{\infty}a_k z^k, g(z)=\sum_{k=1}^{\infty}b_k z^k \quad a_1>0, a_k\geq 0 \text{ and } b_k\geq 0 \tag{4}$$

is said to be in the family $TS^*_{H,\lambda}(z_0)$ satisfying the condition (4) and

$$f(z_0)=z_0, -1<z_0<1, z_0\neq 0 \tag{5}$$

where $a_1 z_0-\sum_{k=2}^{\infty}a_k z_0^k+\sum_{k=1}^{\infty}b_k z^k=z_0$ i.e. $a_1=1+\sum_{k=2}^{\infty}a_k z_0^k-\sum_{k=1}^{\infty}b_k z_0^{k-1}$.

In this paper, we investigate sense preserving harmonic univalent function of form (4) with a fixed point (5) and determine the coefficients bounds, distortion and extreme points, convolution convex combination and a family of class preserving integral operator.

2. Fractional Calculus

The following definitions of fractional derivatives and fractional integrals are due to Owa [5] and Srivastava and Owa [9].

Definition 2.1. The fractional derivative of order λ is defined, for a function $f(z)$ of the form (2), by

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\varsigma)}{(z-\varsigma)^{\lambda}}d\varsigma, \tag{6}$$

where $0\leq\lambda<1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin.

Definition 2.2. Under the hypothesis of Definition 2.1, the fractional derivative of order $n+\lambda$ is defined for a function $f(z)$ by

$$D_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}D_z^{\lambda}f(z) \tag{7}$$

where $0\leq\lambda<1$, and $n\in N_0=\{0,1,2.....\}$.

Using the Definition 2.2 and its known extreme involving fractional derivatives Owa and Srivastava [10] introduced the operator $\Omega^{\lambda}:A\to A$ as follows

$\Omega^{\lambda}f(z)=\Gamma z^{\lambda}D_z^{\lambda}f(z)$ $(\lambda\neq 2,3,4)$ where A denote the class of function of the (3) which are analytic in U.

Let $TS^*_{H,\lambda}(\alpha,t,z_0)$ denote the subclass of S_H consisting of function f of the form (4) satisfy the following condition

$$\Re\left\{\frac{z(\Omega^{\lambda}h(z))'-\overline{z(\Omega^{\lambda}g(z))'}}{(1-t)a_1 z+t(\Omega^{\lambda}h(z)+\overline{\Omega^{\lambda}g(z)})}\right\}>\alpha, \tag{8}$$

with condition (5), where $0\leq\alpha<1, 0\leq\lambda<1$ and $0\leq t\leq 1$.

We note that for $t=1$ the class $S^*_{H,\lambda}(\alpha,t)$ reduces to $S^*_{H,\lambda}(\alpha)$ studied by Dixit and Porwal [5] and $\lambda=0, t=1$ the class $S^*_{H,\lambda}(\alpha,t)$ and $TS^*_{H,\lambda}(\alpha,t)$ reduces to the class studied by Jahangiri [5].

3. **Main Results**

We begin with a sufficient coefficient condition for function in $TS^*_{H,\lambda}(\alpha,t,z_0)$.

Theorem 3.1. Let $f=h+\overline{g}$ be such that h and g are given by (4).

Then $f\in TS^*_{H,\lambda}(\alpha,t,z_0)$, iff

$$\left(\sum_{k=2}^{\infty}\frac{k-\alpha t}{1-\alpha}|a_k|+\sum_{k=1}^{\infty}\frac{k+\alpha t}{1-\alpha}|b_k|\right)\phi(k,\lambda)\leq a_1 \tag{9}$$

where $0\leq\alpha<1, 0\leq\lambda<1, 0\leq t\leq 1$

and

$$\phi(k,\lambda)=\frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)}.$$

Proof. First we note that *f* is locally univalent and sense-preserving in *U*. This is because

$$|h'(z)|\geq a_1-\sum_{k=2}^{\infty}k|a_k|r^{k-1}>a_1-\sum_{k=2}^{\infty}\frac{k-\alpha t}{1-\alpha}\phi(k,\lambda)|a_k|$$

$$\geq\sum_{k=1}^{\infty}\frac{k+\alpha t}{1-\alpha}\phi(k,\lambda)|b_k|\geq\sum_{k=1}^{\infty}k|b_k|>\sum_{k=1}^{\infty}k|b_k|r^{k-1}\geq|g'(z)|.$$

To show that *f* if univalent in *U*, suppose $z_1,z_2\in U$ such that $z_1\neq z_2$ then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_2)-h(z_2)}\right|\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty}b_k(z_1^k-z_2^k)}{a(z_1-z_2)+\sum_{k=2}^{\infty}a_k(z_1^k-z_2^k)}\right|$$

$$>1-\frac{\sum_{k=1}^{\infty}k|b_k|}{a_1-\sum_{k=2}^{\infty}k|a_k|}$$

$$\geq 1-\frac{\sum_{k=1}^{\infty}\frac{k+\alpha t}{1-\alpha}\phi(k\lambda)|b_k|}{a_1-\sum_{k=2}^{\infty}\frac{k+\alpha t}{1-\alpha}\phi(k,\lambda)|a_k|}$$

$\geq 0.$

Now we show that $f\in TS^*_{H,\lambda}(\alpha,t,z_0)$ using the fact that $\Re(w)\geq\alpha$ iff $|1-\alpha+w|\geq|1+\alpha-w|$ it is suffices to show that

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|\geq 0\,, \qquad (10)$$

Where

$$A(z)=z(\Omega^\lambda h(z)'-\overline{z(\Omega^\lambda g(z))'}$$

$$B(z)=(1-t)a_1z+t(\Omega^\lambda h(z)+\overline{z(\Omega^\lambda g(z))}$$

Substituting the values of *A(z)* and *B(z)* in L.H.S. of (10)

$$|A(z)-(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$$

$$=|z(\Omega^\lambda h(z))'-\overline{z(\Omega^\lambda g(z))'}+(1-\alpha)\{(1-t)a_1z+t(\Omega^\lambda h(z)+\overline{\Omega^\lambda g(z)}\}|$$

$$-|(z\Omega^\lambda h(z))'-\overline{z(\Omega^\lambda g(z))'}-(1+\alpha)\{(1-t)a_1z+t(\Omega^\lambda h(z)+\overline{\Omega^\lambda g(z)}\}|$$

$$\left|(2-\alpha)a_1z+\sum_{k=2}^{\infty}(k+t-\alpha t)\phi(k,\lambda)a_kz^k-\sum_{k=1}^{\infty}\overline{(k-t+\alpha t)\phi(k,\lambda)b_kz^k}\right|$$

$$-\left|-a_1\alpha z+\sum_{k=2}^{\infty}(k-t-\alpha t)\phi(k,\lambda)a_kz^k-\sum_{k=1}^{\infty}\overline{(k+t+\alpha t)\phi(k,\lambda)b_kz^k}\right|$$

$$\geq(2-\alpha)a_1|z|-\sum_{k=2}^{\infty}(k-t-\alpha t)\phi(k,\lambda)|a_k||z^k|-\sum_{k=1}^{\infty}(k-t+\alpha t)\phi(k,\lambda)|b_k||z|^k$$

$$-\alpha a_1|z|-\sum_{k=2}^{\infty}(k-t-\alpha t)\phi(k,\lambda)|a_k||z|^k-\sum_{k=1}\infty^{(k+t+\alpha t)}\phi(k,\lambda)|b_k||z|^k$$

$$=2(1-\alpha)a_1|z|\left\{1-\sum_{k=2}^{\infty}\frac{k-\alpha t}{1-\alpha}\phi(k,\lambda)|a_k||z|^{k-1}-\sum_{k=1}^{\infty}\frac{k+\alpha t}{1-\alpha}\phi(k,\lambda)|b_k||z|^{k-1}\right\}$$

$$>2(1-\alpha)a_1|z|\left\{1-\left(\sum_{k=2}^{\infty}\frac{k-\alpha t}{1-\alpha}\phi(k,\lambda)|a_k|+\sum_{k=1}^{\infty}\frac{k+\alpha t}{1-\alpha}\phi(k,\lambda)|b_k|\right)\right\}$$

≥ 0 using (9).

For ``only if'' part we notice that condition

$$\Re\left\{\frac{z(\Omega^\lambda h(z))'-\overline{z(\Omega^\lambda g(z))'}}{(1-t)z+t(\Omega^\lambda h(z)+\overline{(\Omega^\lambda g(z))}}\right\}\geq\alpha \text{ is equivalent to}$$

$$\Re\left\{\frac{(1-\alpha)a_1z-\sum_{k=2}^{\infty}(k-\alpha t)\phi(k,\lambda)|a_k|z^{k-1}-\sum_{k=1}^{\infty}(k+\alpha t)\phi(k,\lambda)|b_k|z^{k-1}}{a_1z-\sum_{k=2}^{\infty}|a_k|t\phi(k,\lambda)z^k+\sum_{k=1}^{\infty}|b_k|t\phi(k,\lambda)\overline{z}^k}\right\}\geq 0.$$

The above condition must hold for all values of $z, |z| = r < 1$. Upon choosing the value of z on the positive real axis $0 \le z = r < 1$ we must have

$$\frac{a_1(1-\alpha) - \sum_{k=2}^{\infty}(k-\alpha t)\phi(k,\lambda)\,|a_k|\,r^{k-1} - \sum_{k=1}^{\infty}(k+\alpha t)\phi(k,\lambda)\,|b_k|\,r^{k-1}}{1 - \sum_{k=2}^{\infty}|a_k|\,t\phi(k,\lambda)r^{k-1} + \sum_{k=1}^{\infty}|b_k|\,t\phi(k,\lambda)r^{k-1}} \ge 0. \qquad (11)$$

If the condition (9) does not hold then the numerator in (11) is negative for r sufficiently close to 1.

Thus there exists $z_0 = r_0$ in (0,1) for which quotient in (11) is negative. This contradict the required condition for $f \in TS^*_{H,\lambda}(\alpha,t,z_0)$, and so proof is complete.

Next, we determine the extreme point, closed convex hulls of $TS^*_{H,\lambda}(\alpha,t,z_0)$ denoted by clco $TS^*_{H,\lambda}(\alpha,t,z_0)$.

Theorem 3.2. If $f \in clcoTS^*_{H,\lambda}(\alpha,t,z_0)$ iff $f(z) = \sum_{k=1}^{\infty}\{x_k h_k(z) + y_k g_k(z)\}$ where,

$$h_1(z) = z, h_k(z) = z - \frac{(1-\alpha)}{(k-\alpha t)\phi(k,\lambda)} z^k, \quad g_k(z) = z - \frac{(1-\alpha)}{(k+\alpha t)\phi(k,\lambda)} \overline{z}^k ,$$

$$x_k \ge 0, y_k \ge 0, \quad \sum_{k=1}^{\infty}(x_k + y_k) = a_1.$$

In particular extreme points of $TS^*_{H,\lambda}(\alpha,t,z_0)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For the function of the form

$$f(z) = \sum_{k=1}^{\infty}\{x_k h_k(z) + y_k g_k(z)\}.$$

Put the values of $h_k(z)$ and $g_k(z)$

$$= \sum_{k=1}^{\infty}(x_k + y_k) z a_1 - \sum_{k=2}^{\infty}\frac{(1-\alpha)}{(k-\alpha t)\phi(k,\lambda)} x_k z^k + \sum_{k=1}^{\infty}\frac{(1-\alpha)}{(k+\alpha t)\phi(k,\lambda)} y_k \overline{z}^k.$$

Then,

$$\sum_{k=2}^{\infty}\left(\frac{k-\alpha t}{1-\alpha}\right)\phi(k,\lambda)\left\{\frac{(1-\alpha)}{(k-\alpha t)\phi(k,\lambda)}\right\}x_k$$

$$+\sum_{k=1}^{\infty}\left(\frac{k+\alpha t}{1-\alpha}\right)\phi(k,\lambda)\left\{\frac{(1-\alpha)}{(k+\alpha t)\phi(k,\lambda)}\right\}y_k$$

$$= \sum_{k=2}^{\infty}x_k + \sum_{k=1}^{\infty}y_k$$

$$= a_1 - x_1 \le a_1$$

and so $f \in TS^*_{H,\lambda}(\alpha,t,z_0)$.

Conversely, suppose that $f \in clcoTS^*_{H\lambda}(\alpha,t,z_0)$. Set

$$x_k = \left(\frac{k-\alpha t}{1-\alpha}\right)\phi(k,\lambda)\,|a_k| \quad (k = 2,3,4...) \text{ and}$$

$$y_k = \left(\frac{k+\alpha t}{1-\alpha}\right)\phi(k,\lambda)\,|b_k| \quad (k = 1,2,3)$$

$0 \le x_k \le a_1$ $(k = 2,3,4...)$ and $0 \le y_k \le a_1$ $(k = 1,2,3)$.

Then we define

$$x_1 = a_1 - \sum_{k=2}^{\infty}x_k - \sum_{k=1}^{\infty}y_k, \; x_1 \ge 0 \text{ consequently we obtain}$$

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_k(z)) \text{ as we require.}$$

The following theorem gives the bounds for function in $TS^*_{H,\lambda}(\alpha,t,z_0)$ which yields a covering result for this class.

Theorem 3.3. $f \in TS^*_{H\lambda}(\alpha,t,z_0)$ then $|z| = r < 1$,

$$|f(z)| \le (|a_1| + |b_1|)r + \left(\frac{1-\alpha}{2-\alpha t}|a_1| - \left(\frac{1+\alpha t}{2-\alpha t}\right)|b_1|\right)\frac{2-\lambda}{2}r^2,$$

and

$$|f(z)| \ge (1 - |b_1|)r - \left(\frac{1-\alpha}{2-\alpha t}|a_1| - \left(\frac{1+\alpha t}{2-\alpha t}\right)|b_1|\right)\frac{2-\lambda}{2}r^2.$$

Proof. We only prove the right hand inequality. The proof of left hand inequality is similar and will be omitted. Let $f \in TS^*_{H\lambda}(\alpha,t,z_0)$ taking the absolute value of f we obtain

$$|f(z)| \le (|a_1| + |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k$$

$$\le (|a_1|+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2$$

$$\le (|a_1|+|b_1|)r+\frac{(1-\alpha)}{(2-\alpha t)\phi(2,\lambda)}\sum_{k=2}^{\infty}\frac{(2-\alpha t)\phi(2,\lambda)}{(1-\alpha)}(|a_k|+|b_k|)r^2$$

$$= (|a_1|+|b_1|)r+\frac{(1-\alpha)}{(2-\alpha t)\phi(2,\lambda)}\sum_{k=2}^{\infty}\frac{(2-\alpha t)\phi(2,\lambda)}{(1-\alpha)}(|a_k|+|b_k|)r^2$$

$$\le (|a_1|+|b_1|)r+\frac{(1-\alpha)\Gamma(3-\lambda)}{(2-\alpha t)\Gamma 3\Gamma(2-\lambda)}\sum_{k=2}^{\infty}\left(\frac{k-\alpha t}{1-\alpha}|a_k|+\left(\frac{k+\alpha t}{1-\alpha}\right)|b_k|\right)r^2$$

$$\le (|a_1|+|b_1|)r+\frac{(1-\alpha)(2-\lambda)}{2(2-\alpha t)}\sum_{k=2}^{\infty}\left(|a_1|-\left(\frac{1+\alpha t}{1-\alpha}\right)|b_1|\right)r^2$$

$$= (|a_1|+|b_1|)r+\left(\left(\frac{1-\alpha}{2-\alpha t}\right)|a_1|-\frac{1+\alpha t}{2-\alpha t}|b_1|\right)\frac{2-\lambda}{2}r^2.$$

4. Convolution and Convex Combination

In this section we show that the class $TS^{*}_{H,\lambda}(\alpha,t,z_0)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions.

Let *f(z)* and *F(z)* be defined by

$$f(z)=a_1z-\sum_{k=2}^{\infty}|a_k|z^k+\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$

$F(z)=A_1z-\sum_{k=2}^{\infty}|A_k|z^k+\sum_{k=1}^{\infty}|B_k|\overline{z}^k$ we define the above convolution of two harmonic functions *f* and *F* as

$$(f*F)(z)=f(z)*F(z)=a_1A_1z-\sum_{k=2}^{\infty}|a_kA_k|z^k+\sum_{k=1}^{\infty}|b_kB_k|\overline{z}^k. \qquad (12)$$

Using this definition, we show that the class $TS^{*}_{H,\lambda}(\alpha,t,z_0)$ is closed under convolution.

Theorem 4.1. $0\le\beta\le\alpha<1$ let $f\in TS^{*}_{H\lambda}(\beta,t,z_0)$ and $F\in TS^{*}_{H,\lambda}(\alpha,t,z_0)$

where

$$f(z)=a_1z-\sum_{k=2}^{\infty}|a_k|z^k+\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$

and

$$F(z)=A_1z-\sum_{k=2}^{\infty}|A_k|z^k+\sum_{k=1}^{\infty}|B_k|\overline{z}^k$$

Then

$(f*F)\in T^{*}_{H\lambda}(\alpha,t,z_0)\subset T^{*}_{SH\lambda}(\beta,t,z_0)$

Proof. The proof of this theorem is much akin to that of corresponding results of [5], therefore we omit the details involved.

Theorem 4.1. The class $TS^{*}_{H,\lambda}(\alpha,t,z_0)$ is closed under convex combination.

Proof. The proof of this theorem is much akin to that of corresponding results of [5], therefore we omit the details involved.

5. A family of class presenting integral operator

Let $f(z)=h(z)+g(\overline{z})$ be defined by (1)

Let us define *f(z)* by the relation

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)d(t)+\overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt},\ \ c>-1. \qquad (13)$$

The proof of the following theorem is similar to that of corresponding results of [5], therefore we only state the results.

Theorem 5.1. Let $f(z)=h(z)+g(\overline{z})$ be given by (1) and $f\in TS^{*}_{H\lambda}(\alpha,t,z_0), 0\le\lambda<1$. Then *F(z)* defined by (13) is also in the class $TS^{*}_{H\lambda}(\alpha,t,z_0)$.

References

[1] Choi J. Ho (2005), On a subclass of analytic functions with negative coefficients involving the Owa-Srivastava fractional calculus operator. Indian J. Pure Appl. Math., 36 (5), 251-253.

[2] Clunie J. and Sheil-Small T. (1984), Harmonic univalent functions, Ann. Acad. Sci. Fen. Series AI Math., 9 (3), 3-25.

[3] Kumar Vinod, Porwal Saurabh and Dixit and Poonam (2010), A new subclass of harmonic univalent functions defined by fractional calculus, Indian J. Math., 52 (3), 599-613.

[4] and Pathak A.L. (2005), On a new class of close to convex functions with positive coefficients, Bull. Cal. Math., Soc. 97(6), 531-542.

[5] Dixit K.K.and Porwal Saurabh (2010), An application of fractional calculus to harmonic univalent functions, Bull. Cal. Math., Soc., 102 (4), 343-352.

[6] Jahangiri J.M. (1999), Harmonic functions starlike in the unit disc, J. Math., Anal. Appl., 235, 470-477.

[7] Kulkarni S.R. and Naik, U.K (1996)., A new class of univalent functions with negative coefficient with fixed point, Indian J. Pure. Appl. Math., 27 (7), 681-688.

[8] Mishra A.K. and Gochhayat P. (2006), Applications of the Owa-Srivastava operator to the class of k-uniformly convex functions, Eract. Calc. Appl. Anal. 9 (4), 323-331.

[9] Owa S. (1978), On the distortion theorem I, Kyungpook Math., J., 18, 53-59.

[10] Owa S. and Srivastava H.M. (1987), Univalent and starlike generalized hypergeometric functions, Canad. J. Math., 39, 1057-1077.

[11] Silverman H(1976)., Extreme points of univalent functions with two fixed points, Trans Amer. Math., Soc., 219, 387-395.

[12] Silverman H. (1998), Harmonic univalent functions with negative coefficients, J. Math., Anal. Appl., 220, 283-289.

[13] Silverman H. and Silvia E.M. (1999), Subclasses of harmonic univalent functions, New Zealand J. Math., 28, 275-284.

[14] Srivastava H.M. and Owa S. (1984), An application of the fractional derivative, Math., Japon., 29, 383-389.

[15] Srivastava H.M. and Mishra A.K. (2000), Applications of fractional calculus to parabolic starlike and uniformly convex functions, Comput. Math., Appl., 39, 57-69.