K-Distance Non-negative Signed Domination Number of Graphs

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Abstract : Let G be a finite and simple graph with the vertex set V = V(G) and edge set E = E(G). If v is a vertex of a graph G, the open k-neighborhood of v, denoted by $N_k(v)$ and $N_k[v] = N_k(v) \cup \{v\}$ is the closed k-neighborhood of v. A function $f : V(G) \to \{-1, +1\}$ is a k-distance non-negative signed dominating function (k-DNNSDF) of a graph G, if for every vertex $v \in V$, $f(N_k[v]) = \sum_{u \in N_k[v]} f(u) \geq 0$. The k-distance non-negative signed domination number (k-DNNSDN) of a graph G equals the minimum weight of a k-DNNSDF of G, denoted by $\gamma_{k,s}^{NN}(G)$. This paper contains some properties of k-DNNSDN in graphs and some families of graphs such as cycles, paths, Complement of cycles, Complete graphs, Wheel graphs and Friendship graphs which admit 2-DNNSDF.

IndexTerms - **Signed dominating function, k-distance non-negative signed dominating function**.

I. INTRODUCTION

Let G be a finite and simple graph with the vertex set V = V(G) and edge set E = E(G). If v is a vertex of a graph G, the open k-neighborhood of v, denoted by $N_k(v)$ and $N_k[v] = N_k(v) \cup \{v\}$ is the closed k-neighborhood of v. $\delta_k(G) = \min\{|N_k(v)| v \in V\}$ and $\Delta_k(G) = \max\{|N_k(v)| v \in V\}$.

In 1995, J.E. Dunbar et al. defined signed dominating function. A function $f : V \to \{-1, +1\}$ is a signed dominating function of G, if for every vertex $v \in V$, $f(N[v]) \geq 1$. The signed domination number, denoted by $\gamma_s(G)$, is the minimum weight of a signed dominating function on G [1].

In 2013 [2], Zhongsheng Huang et al. introduced the concept of on non-negative signed domination in graphs. A function $f : V \to \{-1, +1\}$ is a non-negative signed dominating function of G, if for every vertex $v \in V$, $f(N[v]) \geq 0$. The non-negative signed domination number, denoted by $\gamma_s^{NN}(G)$. is the minimum weight of a non-negative signed dominating function on G.

In this paper, we introduced the concept of k-distance non-negative signed dominating function. A function $f : V(G) \to \{-1, +1\}$ is a k-distance non-negative signed dominating function (k-DNNSDF) of a graph G, if for every vertex $v \in V$, $f(N_k[v]) = \sum_{u \in N_k} f(u) \geq 0$. The k-distance non-negative signed domination number (k-DNNSDN) of a graph G equals the minimum weight of a k-DNNSDF of G, denoted by $\gamma_{k,s}^{NN}(G)$. This paper contains some properties of k-DNNSDN in graphs and some families of graphs such as cycles, paths, Complement of cycles, Complete graphs, Wheel graphs and Friendship graphs which admit 2-DNNSDF.

MAIN RESULTS

In this section, we obtain some properties of k-DNNSDN in graphs.

Lemma .1. *Let* **f** *be a k-DNNSDF of G and let $S \subset V$. Then $f(S) \equiv |S| (mod\ 2)$.*

Proof. Let $S^+ = \{v | f(v) = 1, v \in S\}$ and $S^- = \{v | f(v) = -1, v \in S\}$. Then $|S^+| + |S^-| = |S|$ and $|S^+| - |S^-| = f(S)$. Therefore $f(S) + |S| = 2|S|$.

Theorem .1. *Let G be a graph of order n. If $\gamma_{k,s}^{NN}(G) = n$, then $G \approx \overline{K_n}$.*

Proof. Proof. Let $\gamma_{k,s}^{NN}(G) = n$. If $\deg(v) \geq 1$ for some $v \in V(G)$, then the function $f : V(G) \to \{-1, +1\}$ defined by $f(v) = -1$ and $f(x) = +1$ for all other vertices x, is k-DNNSDF and this implies that $\gamma_{k,s}^{NN}(G) \leq n - 2$, a contradiction. Thus $\Delta(G) = 0$ and so $G \approx \overline{K_n}$.

Observation .2.1. *Let G be a graph of order n and k be a positive integer. Then $\gamma_{k,s}^{NN}(G) = \gamma_s^{NN}(G^k)$.*

Proof. Let f be a k-DNNSDF of G. It is easy to see that for every $v \in V(G)$, $N_k[v] = N_{G^k}[v]$. Hence $f(N_{G^k}[v]) = f(N_k[v])$ Therefore f is a k-DNNSDF of G if and only if f is a k-distance non-negative signed dominating set of G^k. Thus $\gamma_{k,s}^{NN}(G) = \gamma_s^{NN}(G^k)$.

Lemma .2. *Let G be a graph of order n. Then* $2\gamma(G) - n \leq \gamma_s^{NN}(G)$.

Proof. Let f be a minimum non-negative signed dominating function of G. Let $V^+ = \{u \in V : f(u) = +1\}$ and $V^- = \{u \in V : f(u) = -1\}$. If $v \in V^-$ since $f(N_G[v]) \geq 0$, then v has at least one neighbor in V^+. Therefore $V^+$is a dominating set for G and $|V^+| \geq \gamma(G)$. Since $\gamma_s^{NN}(G) = |V^+| - |V^-|$ and $n = |V^+| + |V^-|$, then $\gamma_s^{NN}(G) = 2|V^+| - n$ and finally we have $\gamma_s^{NN}(G) \geq 2\gamma(G) - n$.

Lemma .3. *Let $n \geq 5$ be an integer. Then the cycle C_n admits 2-DNNSDF with*

$\gamma_{2,s}^{NN}(C_n) \leq k$ when $n = 5k$.

$\gamma_{2,s}^{NN}(C_n) \leq k + 1$ when $n = 5k+1$.

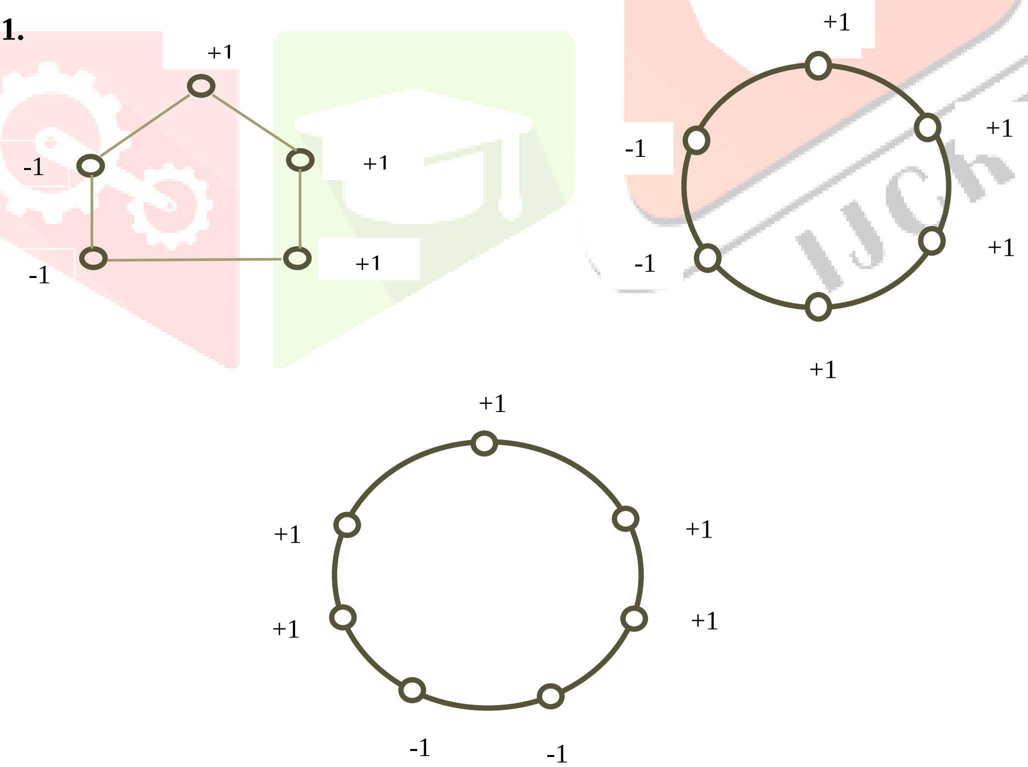
$\gamma_{2,s}^{NN}(C_n) \leq k + 2$ when $n = 5k + 2$ or $n = 5k + 4$.

$\gamma_{2,s}^{NN}(C_n) \leq k + 3$ when $n = 5k+3$. $a_{i \oplus n}$

Proof. Let $n \geq 5$ be an integer. Let $V(C_n) = \{a_i / 1 \leq i \leq n\}$ and $E(C_n) = \{a_i \, a_{i \oplus_n} 1 / 1 \leq i \leq n\}$. Define a function $f : V(C_n) \rightarrow \{-1, +1\}$ such that $f(a_i) = -1$, *when* $i = 5l$ *or* $i = 5l - 1, l \geq 1$ and otherwise $f(a_i) = +1$. Consider the vertex a_i for $1 \leq i \leq n$, $N_2[a_i] = \{a_{i-2}, a_{i-1}, a_i, a_{i+1}, a_{i+2}\}$. From the above labeling, it is easy to observe that at least 3 vertices of any five consecutive vertices must have +1 sign and hence $f(N_2[a_i]) \geq 1$ for all i, $1 \leq i \leq n$.

Thus from the above labeling the result follows.

Example.1.



From the above graphs we observe that $\gamma_{2,s}^{NN}(C_5) \leq 1 \neq 3 = n - 2$, $\gamma_{2,s}^{NN}(C_6) \leq 2 \neq 4 = n - 2$ and $\gamma_{2,s}^{NN}(C_7) \leq 3 \neq 5 = n - 2$. From Lemma .3 and Example .1, we can have following result.

Remark .1. For $n \geq 8$, $\gamma_{2,s}^{NN}(C_n) \leq \lceil n/5 \rceil + 3 < n - 2$.

Lemma .4. *Let $n \geq 5$ be an integer. Then the path P_n admits 2-DNNSDF with*

$\gamma_{2,s}^{NN}(P_n) \leq$ k when n = 5k or n = 5k + 2.

$\gamma_{2,s}^{NN}(P_n) \leq$ k + 1 when n = 5k + 3.

$\gamma_{2,s}^{NN}(P_n) \leq$ k + 2 when n = 5k + 1 or n = 5k + 4.

Proof. Let $V(P_n) = \{a_i / 1 \leq i \leq n\}$ and $E(P_n) = \{a_i a_{i+1} / 1 \leq i \leq n-1\}$.
Case 1: Suppose n = 5k or n = 5k + 3 or n = 5k + 4 for k ≥ 1. A 2-DNNSDF f on P_n is given by $: V(P_n) \to \{-1, +1\}$ define by $f(a_i) = -1$, $when\ i = 5l\ or\ i = 5l - 4, 1 \leq l \leq k$ and otherwise $f(a_i) = +1$.
Case 2: Suppose n = 5k + 1 or 5k + 2 for k ≥ 1.
A 2-DNNSDF f on P_n is given by $f : V(P_n) \to \{-1, +1\}$ define by $f(a_i) = -1$, $when\ i = 5l\ or\ i = 5l - 4, 1 \leq l \leq k$, $f(a_i) = +1$ when i = 5k + 1 or 5k + 2 and otherwise $f(a_i) = +1$ Consider the vertex a_i for 3 ≤ i ≤ n − 2, $N_2[a_i] = \{a_{i-2}, a_{i-1}, a_i, a_{i+1}, a_{i+2}\}$. From the above labeling, it is easy to observe that at least 3 vertices of $N_2[a_i]$ must have +1 sign and hence $f(N_2[a_i]) \geq 1$for all i, 3 ≤ i ≤ n − 2. Also the first and last four vertices have at least two vertices of +1 sign. Hence $f(N_2[a_i]) \geq 0$ when i = 2, n − 2.Also the first and last three vertices have at least two vertices of +1 sign. Hence $f(N_2[a_i]) \geq 1$ when i = 1, n.

Thus from the above labeling the result follows.

Example .2.

From the above graphs we observe that $\gamma_{2,s}^{NN}(P_5) \leq 1 \neq 3 = n - 2$, $\gamma_{2,s}^{NN}(P_6) \leq 2 \neq 4 = n - 2$.

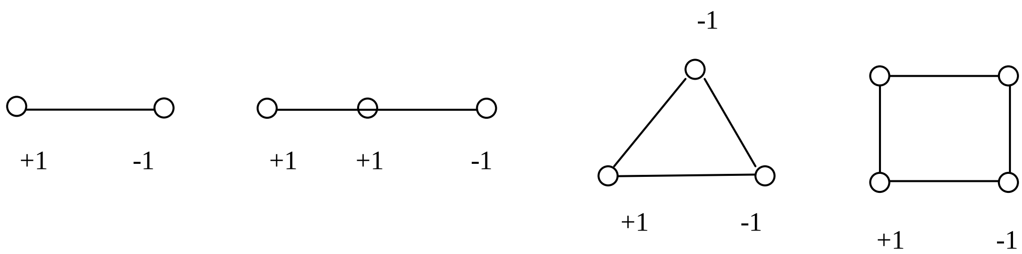
From Lemma .3 and Example .2, we can have following result.

Remark .2. For n ≥ 7, $\gamma_{2,s}^{NN}(P_n) \leq \lceil n/5 \rceil + 32 < n - 2$.

Lemma .5. *Let G be a connected graph of order n. Then $\gamma_{2,s}^{NN}(G) = n - 2$ if and only if*

$G \approx P_2$, P_3 or C_3.

Proof. Let $\gamma_{2,s}^{NN}(G) = n - 2$. We claim that $\Delta(G) \leq 2$. Assume, to the contrary, that $\Delta(G) \geq 3$. Let v be a vertex of maximum degree and let $N_2(v) = \{v_1, \ldots, v_\Delta\}$. If $N_2[v_i] \cap N_2[v_j] = \{v\}$ for some i = j, then define $f : V(G) \to \{-1, +1\}$ by $f(v_i) = f(v_j) = -1$ and f (x) = 1for all other vertices x. Clearly, f is a 2-DNNSDF of G with weight n − 4 which leads to a contradiction. Assume that $N_2[v_i] \cap N_2[v_j] = \{v\}$for every pair i, j, 1 ≤ i = j ≤ Δ(G). It is easy to see that the function $f: V(G) \to \{-1, +1\}$ defined by $f(v) = f(v_1) = -1$ and f (x) = 1 for all other vertices x, is a 2-DNNSDF of G of weight n − 4 which leads to a contradiction. Therefore Δ(G) ≤ 2 and so G is a path or cycle. By Remark .1 and .2, that is not possible to $\gamma_{2,s}^{NN}(G) = n - 2$.



Note that for the graphs C_4 and P_4, we have $\gamma_{2,s}^{NN}(C_4) = \gamma_{2,s}^{NN}(P_4) = 0 \neq n - 2$. Therefore$P_2$, P_3 and C_3 are the only graphs in which $\gamma_{2,s}^{NN}(G) = n - 2$. The graphs P_2, P_3 and C_3 admit k-DNNSDF with $\gamma_{2,s}^{NN}(P_2) = 0$, $\gamma_{2,s}^{NN}(P_3) = 1$ and $\gamma_{2,s}^{NN}(C_3) = 1$.

Lemma .6. *Let n ≥ 5 be an integer. Then the graph C_n^+ admits 2-DNNSDF with $\gamma_{2,s}^{NN}(C_n^+) \leq 0$.*

Proof. Let $V(C_n^+) = \{a_i, b_i / 1 \leq I \leq n\}$ and $E(C_n^+) = \{a_i a_{i+1} / 1 \leq I \leq n-1\} \cup \{a_1 a_n\} \cup \{a_i b_i / 1 \leq I \leq n\}$. Define a function $f : V(C_n^+) \rightarrow \{-1, +1\}$. $f(a_i) = +1$ and $f(b_i) = -1$ for $1 \leq i \leq n$. Now we consider the vertices a_i. $N_2[a_i] = \{a_{i-2}, a_{i-1}, a_i, a_{i+1}, a_{i+2}, b_{i-1}, b_i, b_{i+1}\}$, by the above labeling $f(N_2[a_i]) = 2$ for$1 \leq i \leq n$. Next, we consider the vertices b_i. $N_2[b_i] = \{b_i, a_{i-1}, a_i, a_{i+1}\}$, by the above labeling $f(N_2[b_i]) = 2$ for $1 \leq i \leq n$. Thus f is 2-DNNSDF with $\gamma_{2,s}^{NN}(C_n^+) \leq 0$.

Theorem .2. *Let $n \geq 5$ be an integer. Then the graph $\overline{C_n}$ admits 2-DNNSDF with $\gamma_{2,s}^{NN}(\overline{C_n}) \leq 0$ when n is even and $\gamma_{2,s}^{NN}(\overline{C_n}) \leq 1$ when is n odd.*

Proof. Let $V(\overline{C_n}) = \{a_i / 1 \leq i \leq n\}$. Define a function $f : V(\overline{C_n}) \rightarrow \{-1, +1\}$ by $f(a_i) = +1$ when is n odd and $f(a_i) = -1$ when n is even for $1 \leq i \leq n$. Note that $N_2[a_i] = V(\overline{C_n})$ for $1 \leq i \leq n$. Suppose n is odd, then by the above labeling $f(N_2[a_i]) = \frac{n+1}{2}(+1) + \frac{n-1}{2}(-1) = 1$. Thus f is 2-DNNSDF with $\gamma_{2,s}^{NN}(\overline{C_n}) \leq 1$. Suppose n is even, then by the above labeling $f(N_2[a_i]) = \frac{n}{2}(+1) + \frac{n}{2}(-1) = 0$. Thus f is 2-DNNSDF with $\gamma_{2,s}^{NN}(\overline{C_n}) \leq 0$.

After studying the above results, we find the following more general result:

Theorem .3. *If diam(G) ≥ k, then G admits k-DNNSDF.*

Proof. Since diam(G) ≥ k, for every vertex $v \in V(G)$, we have $N_k[v] = V(G)$. Suppose $n = 2p$. Then we can label p vertices with +1 signs and p vertices with -1 signs. In this case, $f(N_k[v]) = p(+1) + p(-1) = 0$.
Suppose $n = 2p + 1$. Then we can label $p + 1$ vertices with +1 signs and p vertices with -1 signs. In this case, $f(N_k[v]) = (p + 1)(+1) + p(-1) = 1$. Thus G admits k-DNNSDF.

The next result follows immediately from the above theorem.

Lemma .7. The complete graph K_n admits 2-DNNSDF for n ≥ 1.

For the integers m, n(≥ 1), the complete bipartite graph $K_{m,n}$ admits 2 DNNSDF.
The wheel graph W_n admits 2-DNNSDF for n ≥ 3.
The graph $G = P_m + P_n$ admits 2-DNNSDF for m, n ≥1.
The friendship graph T_n admit 2-DNNSDF.

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