Comparative Study of Von Neumann's regular, Bourne's regular and k-regular on Semi rings

S.Dickson

Assistant Professor in Mathematics, Vivekanandha College for Women, Tiruchengode, Tamilnadu, India.

Abstract: In this paper, a study of Von Neumann's regular, Bourne's regular and k-regular on Semi ring are discussed. Many procedures have been established during the past few decades yet there is need of procedure to compare these types of regulars on semi ring. Here the objective is to compare these three regulars and to give the relationship between them.

Index Terms: Regular, Semi ring, Idempotent.

1. INTRODUCTION

Von Neumann regular rings were introduced by Von Neumann[1] in 1936 under the name of "regular rings". Von Neumann introduced regular in Rings by defining if any element in a ring R is said to be regular if axa = a for some x in R.

In 1934 Vandiver[2] introduced an algebraic system, which consisted of a non empty set S with two binary operations addition (+) and multiplication (\cdot) such that S was a semi group under both operations. The system (S,+, \cdot) satisfied both distributive laws but did not satisfy the cancellation law of addition. The system he constructed was ring-like but not exactly a ring. Vandiver called this system a `semi ring'.

In this paper types of regulars on semi rings which are studied by S. Bourne [3], M.R. Adhikari, M.K.Sen and H.JWeinert[4] have been shortened here for this paper. The relation between these regulars in semi rings and additive idempotent commutative semi rings is explained.

2. PRELIMINARIES

Some important definitions are given below.

Definition 2.1: Semi group

A semi group is a non empty set, G, together with an operation *. To qualify as a semi group, the set and operation, (G, *), must satisfy the following two axioms:

Closure: For all a, b in G, the result of the operation, a * b, is also in G. Associativity: For all a, b and c in G, (a*b)*c = a*(b*c).

Definition 2.2: Idempotent

Let (G, *) be a group. An element $a \in G$ is said to be idempotent if a * a = a. If all the elements in G are idempotent then the group G is called idempotent group under the binary operation.

Definition2.3: Regular Ring

A ring R is said to be *regular* if for each element $a \in R$ there exist some element $x \in R$ such that axa = a.

Example 2.4:

The rings Q, the set of all rational numbers and R, the set of all real numbers are regular rings under usual addition and multiplication.

Definition 2.5: Semi ring

A triple $(S, +, \cdot)$ is called a *semi ring* if (S, +) and (S, \cdot) are semi groups and \cdot is distributive over +.

Example 2.6:

- (1) Z, the set of all integers forms semi ring under usual addition and multiplication.
- (2) N, the set of all natural numbers forms semi ring under usual addition and multiplication.
- (3) Q and R are semi rings under usual addition and multiplication.

3. VON NEUMANN REGULAR ON SEMI RINGS

In this how Von Neumann regular is used in semi rings has been given.

Definition 3.1: Semi ring

It is already defined that a triple $(S, +, \cdot)$ is called a *semi ring* if (S, +) and (S, \cdot) are semi groups and \cdot is distributive over +.

An element $0 \in S$ is called a zero of the semi ring $(S, +, \cdot)$ if x + 0 = 0 + x = x and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$. Note that every semi ring contains at most one zero.

A semi ring $(S, +, \cdot)$ is called additively [multiplicatively]commutative if x + y = y + x [$x \cdot y = y \cdot x$] for all $x, y \in S$. We say that (S,+, *) is commutative if it is both additively and multiplicatively commutative.

Definition 3.2: matrix semi ring

Let $M_n(S)$ be the set of all $n \times n$ matrices over S. Then under the usual addition and multiplication of matrices, $M_n(S)$ is a semi ring. This semi ring is called *matrix semi ring*.

 $M_n(S)$ is also an additively commutative semi ring with zero and the n × n zero matrix over S is the zero of the matrix semi ring $M_n(S)$. For A $\in M_n(S)$ and i, j $\in \{1, 2, ..., n\}$, let A_{ij} be the entry of A in the ith row and jth column.

Notations 3.3:

Denote by Z, Q and R the set of integers, the set of rational numbers and the set of real numbers, respectively. Let $Z_0^+ = \{x \in Z \mid x \ge 0\}$, $Q_0^+ = \{x \in Q \mid x \ge 0\}$ and $R_0^+ = \{x \in R \mid x \ge 0\}$.

Example 3.4

- (1) Under the usual addition and multiplication of real numbers, Z_0^+ , Q_0^+ and R_0^+ are commutative semi rings with zero 0 which are not rings. If n is a positive integer greater than 1, then $M_n(Z_0^+)$, $M_n(Q_0^+)$ and $M_n(R_0^+)$ are additively commutative semi rings with zero which are not multiplicatively commutative. Also, none of them are rings.
- (2) If $S = \{0,1\}$ with 0 + 0 = 0, 0 + 1 = 1 + 0 = 1 + 1 = 1, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$, then (S, +, *) is a commutative semi ring with zero 0 which is not a ring.
- (3) Let S be a non empty subset of R such that min S exists. Define x⊕y = max{x, y} and x⊙y = min{x, y} for all x, y ∈ S. Then (S, ⊕,⊙) is a commutative semi ring having min S as its zero. Also, if S contains more than one element, then (S, ⊕,⊙) is not a ring.

Definition 3.5: Regular semi ring

A semi ring S is said to be Von Neumann regular if for every $x \in S, x = xyx$ for some $y \in S$.

Examples 3.6:

We can see that the semi rings Q_0^+ and R_0^+ are regular Z_0^+ is not regular. Also, the semi rings in Example 2.4,(2) and (3) are regular semi rings.

Theorem 3.7:

Let S be an additively commutative semi ring with zero 0 and n a positive integer. If $M_n(S)$ is a regular semi ring, then so is S.

Proof:

If S is an additively commutative semi ring with zero 0, n is a positive integer and $A \in M_n(S)$ is such that $A_{ij}=0$ for all $i; j \in \{1,...,n\}$ with $(i, j) \neq (1, 1)$,

then for
$$B \in M_n(S)$$
, where $B = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix}$

$$A = ABA \implies A_{11} = (ABA)_{11} = \sum_{k=1}^n (AB)_{1k} A_{k1} = (AB)_{11} A_{11} + 0 + 0 + \dots + 0 = (\sum_{k=1}^n A_{1k} B_{k1}) A_{11}$$

 $= A_{11}B_{11}A_{11} + 0 + 0 + \ldots + 0$ (n-1 zero terms)

 $= A_{11}B_{11}A_{11}$

Similarly we can prove this for all elements in $M_n(S)$. Hence the theorem.

Remark 3.8:

The converse of the theorem 2.7 need not be true for n = 2, For, Let $S = \{0, 1, 2, 3\}$ and (S, \bigoplus, \bigcirc) the semi ring defined in Example 2.4,(3), that is, $x \oplus y = \max\{x, y\}$ and $x \odot y = \min\{x, y\}$ for all $x, y \in S$. Then (S, \oplus, \odot) is regular. Suppose that $M_2(S)$ is a regular semi ring. Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \in M_2(S)$. Then A = ABA for some $B \in M_2(S)$. Then by the definition of \bigoplus and \odot and $1] \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}$ $3 \begin{bmatrix} B_{21} & B_{22} \end{bmatrix}$ $\begin{bmatrix} 1 \odot B_{21} & 1 \odot B_{22} \\ 2 \odot B_{11} \oplus B_{21} & 2 \odot B_{12} \oplus B_{22} \end{bmatrix}$ $1 \odot B_{22}$ and thus ABA = (AB)A $\begin{bmatrix} 1 \odot B_{21} & 1 \odot B_{22} \\ 2 \odot B_{11} \oplus B_{21} & 2 \odot B_{12} \oplus B_{22} \end{bmatrix}$ $1 \odot B_{22}$ $1 \odot B_{21} \oplus 1 \odot B_{22}$ $(2 \odot B_{12} \oplus B_{22}) \odot 2 \quad (2 \odot B_{11} \oplus B_{21}) \odot 1 \oplus (2 \odot B_{12} \oplus B_{22})$ Since A = ABA, it follows that $1 \odot B_{22} = 0$(1) $(2 \odot B_{11} \oplus B_{21}) \odot 1 \oplus (2 \odot B_{12} \oplus B_{22}) = 3$ From (1), $B_{22} = 0$. Since $(2 \odot B_{11} \oplus B_{21}) \odot 1 \le 1$, by (2), we have $2 \odot B_{12} \oplus B_{22} = 3$. But $B_{22} = 0$, so $3 = 2 \odot B_{12} \oplus B_{22}$ $= 2 \odot B_{12} \leq 2$

which is a contradiction. This shows that $M_2(S)$ is not regular, as desired.

Note 3.9:

Let S be an additively commutative semi ring with zero 0 and n a positive integer with $n \ge 3$. Then the semi ring $M_n(S)$ is regular if and only if S is a regular ring.[4]

4. BOURNE REGULAR ON SEMI RINGS

In 1951, Bourne[3] defined a regular on semi ring which is defined as follows.

Definition: 4.1

A Semi ring is said to be Bourne regular if for all $a \in S$ there exists x, $y \in S$ such that

a + axa = aya.

Example: 4.2

- 1. Let $R_0^+ = \{x \in R \mid x \ge 0\}$ and $Q_0^+ = \{x \in Q \mid x \ge 0\}$. These are the Bourne regular semi rings
- 2. If $S = \{0,1\}$ with 0 + 0 = 0, 0 + 1 = 1 + 0 = 1 + 1 = 1, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$, then (S, +, *) is a Bourne regular semi ring.
- 3. Let S be a non empty subset of R such that min S exists. Define $x \oplus y = \max\{x, y\}$ and $x \odot y = \min\{x, y\}$ for all x, $y \in S$. Then (S, \oplus, \odot) is Bourne regular semi ring.

5. K-REGULAR ON SEMI RINGS

Adhikari, Sen and Weinert renamed Bourne's regular as k-regularity with an additive idempotent to distinguish it with the regularity in the sense of Von Neumann.

Thus an additive idempotent commutative semi ring S is k-regular if for all $a \in S$ there are x, $y \in S$ such that a + axa = aya. Since (S, +) is idempotent we add axa + aya to both the sides to get a + a(x + y)a = a(x + y)a. Therefore k-regular is defined in the following way.

Definition 5.1: k-regular semi ring

An additive idempotent commutative semi ring S is *k*-regular if for all $a \in S$ there is $x \in S$ such that a + axa = axa. Example 5.2:

Let D be a distributive lattice. Consider $S = M_2(D)$, the semi ring of 2×2 matrices on D where $D = (N, +, \cdot)$ where N is the set of all positive integers, $a + b = \max\{a, b\}$ and $a.b = \min\{a, b\}$. Then S is k-regular.

Solution:

Clearly D is an additive idempotent semi ring. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S$ where a, b, c, $d \in D$ and let $a \le b \le c \le d$.

Then for
$$X = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in S$$

$$AXA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a.a + b.b & a.c + b.d \\ c.a + d.b & c.c + d.d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a+b & a+b \\ a+b & c+d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} b & b \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} b.a + b.c & b.b + b.d \\ b.a + d.c & b.b + d.d \end{bmatrix}$$

$$= \begin{bmatrix} a+b & b+b \\ a+c & b+d \end{bmatrix}$$

$$= \begin{bmatrix} b & b \\ c & d \end{bmatrix}$$

$$A + AXA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} b & b \\ c & d \end{bmatrix}$$

Hence A + AXA = AXA. This shows that S is k-regular. Note 5.3:

The above example is not a Von Neumann regular semi ring.

Let A =
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
. If possible let there be X = $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ where x,y,z,t $\in N$, such that A = AXA
i.e. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1.x + 1.z + 1.y + 2.t & 1.x + 2.z + 1.y + 2.t \\ 1.x + 1.z + 3.y + 3.t & 2.x + 2.z + 3.y + 4.t \end{bmatrix}$

This implies that

For,

 $\begin{array}{l} 1 = 1.x + 1.z + 1.y + 2.t \\ 2 = 1.x + 2.z + 1.y + 2.t \\ 3 = 1.x + 1.z + 3.y + 3.t \\ 4 = 2.x + 2.z + 3.y + 4.t \\ Now (1) \text{ implies that } 1 = 1 + 1 + 1 + 2.t \implies 1 = 1 + 2.t \implies 2.t = 1 \implies t = 1 \\ \text{But then (4) implies that } 4 = 2.x + 2.z + 3.y + 1 \le 3 \\ \text{This is a contradiction. Hence S is not regular in the sense of Von Neu mann.} \end{array}$

6. RELATION BETWEEN THE REGULARS

Theorem 6.1:

In a semi ring $(S,+,\cdot)$ every Von Neumann regular is Bourne regular and vice versa if S is a group under addition. **Proof:**

Let $(S,+,\cdot)$ be semi ring and let S be a Von Neumann regular. Then for all $a \in S$ there exists $x \in S$ such that a = axa....(1)Let $y \in S$. Then $aya \in S$

Adding aya on both sides of (1) we get a + aya = axa + aya a + aya = a(x+y)a a + aya = aza where $z = x + y \in S$ Hence for all $a \in S$ there are y, $z \in S$ such that a + aya = aza. Hence S is Bourne regular. Let S be a group under addition and let S be a Bourne regular. Then for all $a \in S$ there are x, $y \in S$ such that a + axa = aya(1)

Since $axa \in S$ and S is a group under addition $-axa \in S$ Adding -axa on both sides of (1) we get a + axa - axa = aya - axaa = a(y-x)a (-axa is an additive inverse of axa $\in S$) a = aza where $z = y - x \in S$. Hence S is Von Neumann regular. Theorem 6.2: In an additive idempotent commutative semi ring (i) Every Bourne regular is k-regular. (ii) Every Von Neumann regular is k-regular and vice versa if cancellation laws hold. **Proof:** Let $(S, +, \cdot)$ be an additive idempotent commutative semi-ring. (i) Assume S is Bourne regular. Then for all $a \in S$ there are x, $y \in S$ such that a + axa = ayaAdding axa + aya on both the sides we get a + axa + axa + aya = aya + axa + ayaa + axa + aya = aya + axa + aya((S,+) is idempotent) ((S,+) is commutative) a + axa + aya = axa + aya + aya((S,+) is idempotent) a + axa + aya = axa + ayaa + a(x + y)a = a(x + y)a(Distributive Laws) a + aza = aza where $z = x + y \in S$. Hence S is k-regular. Let S be k-regular and *cancellation laws hold* in S. Then for all $a \in S$ there exists $x \in S$ such that a + axa = axa. Adding aya on both sides we get, a + axa + aya = axa + aya((S,+) is idempotent) a + axa + aya = axa + aya + aya(By R.C.L) a + axa = axa + aya(Distributive Laws) a + axa = a(x+y)aa + axa = aza where $z = x+y \in S$. Hence S is Bourne regular

(ii) Assume S is Von Neumann regular.

Then for all $a \in S$ there exists an $x \in S$ such that a = axaAdding axa on both the sides we get

a + axa = axa + axa

a + axa = axa ((S,+) is idempotent)

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where x \in S. Hence S is k-regular.
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Let S be k-regular and *cancellation laws hold* in S. Then for all $a \in S$ there exists $x \in S$ such that

 $a + axa = axa \Rightarrow a + axa = axa + axa \Rightarrow a = axa$ (By R.C.L)

Hence S is Von Neumann regular.

7. CONCLUSION

In this paper, a study of Von Neumann's regular, Bourne's regular and k-regular on Semi ring are discussed and the relationship between them explained. This comparative study can be considered as a significant improvement of these regulars with the relation and the reverse relation between all of them on semi ring.

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