A STUDY OF CONVOLUTION INTEGRAL EQUATIONS INVOLVING THE GENERAL POLYNOMIALS

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ABSTRACT: In this paper we first solve a double convolution integral equation involving the product of the general class of polynomials.On account of the general nature of our main result, solutions of a large number of other double convolution integral equations involving the product of several useful polynomials can also be obtained as its special cases. The one variable analogue of the main result which is also quite general in nature and of interest in itself has also been given. A special caseof this latter result yields a known formula obtained by Srivastava and Buschman (3).

INTRODUCTION:

A general class of polynomials [1, p. 1, eq. (1)]

$$S_N^M[x] = \sum_{k=0}^{[N/M]} \frac{(-N)_{Mk} A_{N,k} x^k}{k!}, \quad (N=0,1,2,\ldots..) \tag{1.1}$$

where M is an arbitrary positive integer and the coefficient $A_{N,k}(N, k \geq 0)$ are arbitrary nstants real or complex. On suitably specializing the coefficient $A_{N,k}$ $S_N^M[x]$ yields a number of known polynomials as its special cases. These include, among others, Laguerre polynomials, Hermite polynomials and several others [2. pp. 1 58-161].

The two dimensional Laplace transform of a function $f(x, y)$:

$$\bar{f}(p, q) = L\{f(x,y); p, q\} = \int_0^\infty \int_0^\infty e^{-p,x-q,y} f(x,y)dxdy \tag{1.2}$$

The convolution theorem in the theory of two dimensional Laplace transforms

$$L\{\int_0^x \int_0^y f(x-u, y-v)g(u,v)dudv; p, q\} = L\{f(x,y); p, q\}L\{g(x,y); p, q\} \tag{1.3}$$

MAIN RESULT

The convolution integral equation

$$\int_0^x \int_0^y (x-u)^{\rho_1-1}(y-v)^{\rho_2-1} S_{N_1}^{M_1}[(x-u)]^{\rho_1} S_{N_2}^{M_2}[(y-v)]^{\rho_2} f(u,v)dudv = g(x,y) \tag{2.1}$$

has the solution given by

$$f(x,y) = \int_0^x \int_0^y (x-u)^{m-\rho_1-\sigma_1\mu_1-1}(y-v)^{n-\rho_2-\sigma_2\mu_2-1}.$$

$$\sum_{\lambda_1=0}^\infty \frac{B_{\lambda_1}^i (x-u)^{\sigma_1\lambda_1}}{\Gamma(m-\rho_1-\sigma_1\mu_1+\sigma_1\lambda_1)} \sum_{\lambda_2=0}^\infty \frac{B_{\lambda_2}^2 (y-v)^{\sigma_2\lambda_2}}{\Gamma(n-\rho_2-\sigma_2\mu_2+\sigma_2\lambda_2)} \cdot \frac{\partial(m+n)g(u,v)}{\partial u^m \, \partial v^n} dudv \tag{2.2}$$

where min Re $(\rho_1, \rho_{2,}\sigma_1, \sigma_2) > 0$,$m$ and n are positive integers such that Re (Re $(m-\mu_1\sigma_1-\rho_1) > 0$ Re $(n-\mu_2\sigma_2-\rho_2) > 0$,

$$\bar{g}_{1,y^k}(p,0) = \bar{g}_{2,x^c}(0,q) = g_{x^c y^k}^{(c,k)}(0,0) = 0 \text{ for } 0 \leqslant c < m, 0 \leqslant k < n \tag{2.3}$$

and $B_{\lambda_i}^{(i)}$ are given by the recurrence,

$$B_0^{(i)} = \frac{1}{c_{\mu_i}^{(i)}} \text{and for } v_j > 0 \sum_{\lambda_1=0}^{v_i} B_{\lambda_i}^{(i)} C_{v_i+\mu_i-\lambda_i}^{(i)} = 0 \tag{2.4}$$

Or by

$$B_{\lambda_i}^{(i)} = (-1)^{\lambda_i} \left(C_{\mu_i}^{(i)}\right)^{-\lambda_i-1} \det \begin{bmatrix} C_{\mu_i+1}^{(i)} & C_{\mu}^{(i)} & 0 & 0 & \ldots & 0 \\ C_{\mu_i+2}^{(i)} & C_{\mu_i+1}^{(i)} & C_{\mu_i}^{(i)} & 0 & \ldots & 0 \\ \vdots & & & & & \\ C_{\mu_i+\lambda_i}^{(i)} & C_{\mu_i+\lambda_i-1}^{(i)} & & & \ldots & C_{\mu_i+1}^{(i)} \end{bmatrix} \tag{2.5}$$

and

$$C_{k_i}^{(i)} = \frac{(-N_i)_{M_i k_i} \Gamma(\rho_i + \sigma_i k_i) A_{N_i, k_i}}{k_i!} \tag{2.6}$$

$$k_i = 0,1, \ldots, [N_i/M_i], N_i = 0,1,2, \ldots \text{ and } i = 1,2$$

μ_i is the least k, for which $C_{k_i}^{(i)} \neq 0$

Proof. To solve the convolution integral equation (2.1), we take the double Laplace trans form of its both sides. We easily obtain by the definition of two dimensional Laplace trans form and its convolution property the following result:

$$\left[\int_0^\infty \int_0^\infty e^{-px-qy} x^{\rho_1-1} y^{\rho_y-1} S_{N_z}^{M_1}[x^{o_1}] S_{N_2}^{M_2}[y^{o_2}] dxdy\right] \bar{f}(p,q) = \bar{g}(p,q) \tag{2.7}$$

Now expressing the product of the general class of polynomials in terms of series as defined by (1, 1), we easily arrive at the following result after a little simplification

$$\bar{f}(p,q) = p^{\rho_1} q^{\rho_2} \left[\sum_{k_1=0}^{[N_1/M_1]} C_{k_1}^1 p^{-\sigma_1 k_1}\right]^{-1} \cdot \left[\sum_{k_2=0}^{[N_2/M_2]} C_{k_2}^2 q^{-\sigma_2 k_2}\right]^{-1} \bar{g}(p,q) \tag{2.8}$$

Where $C_{k_i}^{(i)} \neq 0$ are defined as in (2.6).

If μ_i denotes the least k_i for which $C_{k_i} \neq 0$, then the series can be reciprocated. Writing

$$\left[\begin{matrix}[N_i/M_i] - \mu_i \\ \eta_i = 0\end{matrix} C_{\mu_i+\eta_i}^{(i)} p^{-\sigma_i \eta_i}\right]^{-1} = \sum_{\lambda_i=0}^{\infty} B_{\lambda_i}^{(i)} p^{-\sigma_i \lambda_i}, i = 1,2 \tag{2.9}$$

The eqn. (2.8) takes the following form:

$$\bar{f}(p,q) = p^{-(m-\rho_1-\sigma_1\mu_1)} q^{-(n-\rho_3-\sigma_2\mu_2)} \sum_{\lambda_1=0}^{\infty} B_{\lambda_1}^1 p^{-\sigma_1\lambda_1}. \sum_{\lambda_3=0}^{\infty} B_{\lambda_2}^2 q^{-\sigma_2\lambda_2} [p^m q^n \bar{g}(p,q)] \tag{2.10}$$

On taking the inverse of the double Laplace transform of both sides of (2.10), using the convolution theorem given by (1.8) in its Right hand side, we arrive at the desired result (2.2) with the help of (1.3).

SPECIAL CASES:

The convolution integral equation

$$\int_0^x (x-u)^{\rho_1-1} S_N^{\mathrm{M}}[(x-u)^\sigma] f(u)\, du = g(x) \tag{3.1}$$

has the solution given by

$$f(x) = \int_0^x (x-u)^{m-p-\sigma_{\mu-1}} \sum_{\lambda=0}^{\infty} \frac{B_\lambda (x-u)^{\sigma\lambda}}{\Gamma(m-\rho-\sigma\mu+\sigma\lambda)} \frac{\partial^m g(u)}{\partial u^m} du \tag{3.2}$$

where min Re, $(\rho, \sigma) > 0$ m and n are positive integers such that Re $(m - \mu\sigma - \rho) > 0$

$$\frac{\partial^r g(u)}{\partial u^r} = g^r(0) = 0 \text{ for } 0 \leqslant r < m \tag{3.3}$$

andB_λ are given by

$$B_0 = \frac{1}{C_u} \text{ and for } v > 0, \sum_{\lambda=0}^{v} B_\lambda C_{v+\mu-\lambda} = 0$$

Or by

$$B_\lambda = (-1)^\lambda C_\mu^{-\lambda-1} \det \begin{bmatrix} C_{\mu+1} & C_\mu & 0 & 0 & \dots 0 \\ C_{\mu+2} & C_{\mu+1} & C_\mu & 0 & \dots 0 \\ C_{\mu+\lambda} & C_{\mu+\lambda-1} & & \dots & C_{\mu+1} \end{bmatrix}$$

and

$$C_k = \frac{(-N)_{Mk}\Gamma(\rho+\sigma k)A_{N,k}}{k!}$$

K=0, 1…,[N/M], N=0,1,2,…,$\mu$is the latest k for which $C_k \neq 0$.
In the above result if we take $\rho = \sigma = M = 1$

$$A_{N,k} = \frac{1}{k!}, S_N^1[x] \to L_N[x],$$

we arrive at a result given by Srivastava and Buschman [3, p. 215] in a slightly different form.

A number of special cases of (3.1) can also be obtained by specializing the parameters of the general class of polynomial S_N^M involved therein [2, pp. 158-161] but we do not record them here for lack of space.

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