CONVOLUTION INTEGRAL EQUATIONS INVOLVING A GENERAL CLASS OF POLYNOMIALS AND THE MULTIVARIABLE H-FUNCTION

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ABSTRACT: In this paper we first solve a convolution integral equation involving product of the general class of polynomials and the H-function of several variables. Due to general nature of the general class of polynomials and the H-function of several variables which occur as kernels in our main convolution integral equation, we can obtain from it solutions of a large number of convolution integral equations involving products of several useful polynomials and special functions as its special cases.

INTRODUCTION:

A general class of polynomials [3, p. 1, eq. (1)]

$$S_N^M[x] = \sum_{k=0}^{[N/M]} \frac{(-N)_{Mk} A_{N,k} x^k}{k!}, \quad (N=0,1,2,\ldots..) \tag{1.1}$$

where M is an arbitrary positive integer and the coefficient $A_{N,k}(N, k \geq 0)$ are arbitrary nstants real or complex. On suitably specializing the coefficient $A_{N,k}$ $S_N^M[x]$ yields a number of known polynomials as its special cases. These include, among others, Laguerre polynomials, Hermite polynomials and several others [7. pp. 1 58-161].

A special case of the H-function of r variables [6, p. 271, eq. (4.1)]

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H_{p,q:\ p_1,q_{1+1};\ldots,\ p_r,q_r+1}^{o,o:\ 1,n_1;\ldots,1,n_r}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix} \left(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p}: & \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q}: & (0,1),\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;(0,1),\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \int_{L_r} \phi_1(\xi_1) \ldots \phi_r(\xi_r) \psi\,(\xi_1,\ldots,\xi_{2r})$$

$$\times \Gamma(-\xi_1) \ldots \Gamma(-\xi_r)\, z_1^{\xi_1} \ldots z_r^{\xi_r} d\xi_1 \ldots d\xi_r, \omega = \sqrt{-1}. \tag{1.2}$$

Or equivalently [5, p. 64, eq. (1.3)]

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = \sum_{k_1,\ldots k_r=0}^{\infty} \phi_1(k_1) \ldots \phi_r(k_r) \psi\,(k_1,\ldots,k_r) \frac{(-z_1)^{k_1}}{k_1!} \ldots \frac{(-z_r)^{k_r}}{k_r!} \tag{1.3}$$

Where

$$\phi_i(k_i) = \frac{\prod_{j=1}^{n_i} \Gamma\left(1 - c_j^{(i)} + \gamma_j^{(i)} k_i\right)}{\prod_{j=1}^{q_i} \Gamma\left(1 - d_j^{(i)} + \delta_j^{(i)} k_i\right) \prod_{j=n_i+1}^{p_i} \Gamma\left(c_j^{(i)} - \gamma_j^{(i)} k_i\right)} (i = 1,\ldots,r) \tag{1.4}$$

$$\psi(k_1,\ldots,k_r) = \left\{\prod_{j=1}^{p} \Gamma\left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} k_i\right) \prod_{j=1}^{q} \Gamma\left(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} k_i\right)\right\}^{-1}. \tag{1.5}$$

For the convergence, existence conditions and other details of the multivariable H-function refer the book [4, pp. 251-253, eqs. (C.2)-(C.8)].

The following property of the Laplace transform [1, p. 131]

$$L\{f^{(n)}(x); s\} = s^n \bar{f}(s) \quad (1.6)$$

Where

$$L\{f(x); s\} = \int_0^\infty e^{-sx} f(x) dx = \bar{f}(s) \quad (1.7)$$

The well-known convolution theorem for Laplace transform

$$L\left\{\int_0^x f(x-u) g(u) du; s\right\} = L\{f(x); s\} L\{g(x); s\} \quad (1.8)$$

holds provided that the various Laplace transforms occurring in (1.8) exist.

MAIN RESULT

The convolution integral equation

$$\int_0^x (x-u)^{\rho-1} S_N^M[-z_{r+1}(x-u)] H \begin{bmatrix} z_1(x-u) \\ \vdots \\ z_r(x-u) \end{bmatrix} f(u) du = g(x) \quad (2.1)$$

has the solution given by

$$f(x) = \int_0^x (x-u)^{l-\rho-\mu-1} \sum_{j=0}^\infty \frac{E_j (x-u)^j}{\Gamma(j+l-\rho-\mu)} g^{(l)}(u) du \quad (2.2)$$

where Re $(l-\rho-\mu) > 0$, Re$(\rho) > 0$

$g^{(l)}(0) = 0$ (i=0,1,... ,l- 1), l being a positive integer and E, is given by the recurrence relation

$$E_0 \lambda_\mu = 1, \ \sum_{t=0}^q E_t \lambda_{q+\mu-t} = 0, \ q = 1,2,3, \ldots \quad (2.3)$$

Or by

$$E_j = (-1)^j (\lambda_\mu)^{-j-1} \det \begin{bmatrix} \lambda_{\mu+1} & \lambda_\mu & 0 & 0 & \cdots & 0 \\ \lambda_{\mu+2} & \lambda_{\mu+1} & \lambda_\mu & 0 & \cdots & 0 \\ \vdots & \vdots & & & & \\ \lambda_{\mu+j} & \lambda_{\mu+j-1} & & & \cdots & \lambda_{\mu+1} \end{bmatrix} \quad (2.4)$$

and μ is least B for which $\lambda_B \neq 0$

$$\lambda_B = (-1)^B \sum_{k_1+\cdots+k_{r+1}=B} \Delta(k_1, \ldots, k_{r+1}) \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_{r+1}^{k_{r+1}}}{k_{r+1}!} \quad (2.5)$$

Where

$$\Delta(k_1, \ldots, k_{r+1}) = \phi_1(k_1) \cdots \phi_{r+1}(k_{r+1}) \psi(k_1, \ldots, k_{r+1})$$
$$\psi(k_1, \ldots, k_{r+1}) = \Gamma(\rho + k_1 + \cdots + k_{r+1}) \quad (2.6)$$

$$\times \left\{\prod_{j=1}^p \Gamma\left(a_j - \sum_{i=1}^r \alpha_j^{(i)} k_i\right) \prod_{j=1}^q \Gamma\left(1 - b_j + \sum_{i=1}^r \beta_j^{(i)} k_i\right)\right\}^{-1} \quad (2.7)$$

$$\phi_i(k_i) = \prod_{j=1}^{n_i} \Gamma\left(1 - c_j^{(i)} + \gamma_j^{(i)} k_i\right) \left\{\prod_{j=n_i+1}^{p_i} \Gamma\left(c_j^{(i)} - \gamma_j^{(i)} k_i\right) \prod_{j=1}^{q_i} \Gamma\left(1 - d_j^{(i)} + \delta_j^{(i)} k_i\right)\right\}^{-1}$$

$(i = 1, \ldots, r)$ (2.8)

and

$$\phi_{r+1}(k_{r+1}) = \begin{cases} (-N)_{Mk_{r+1}+1} A_{N,k_{r+1}}, & 0 \leqslant k_{r+1} \leqslant \left[\frac{N}{M}\right] \\ 0 & k_{r+1} > \left[\frac{N}{M}\right] \end{cases} \quad (2.9)$$

Proof. To solve the convolution integral equation (2.1) we first take the Laplace transform of its both sides, We easily obtain by the definition of Laplace transform and its convolution property stated in (1.8), the following result

$$\left[\int_0^\infty e^{-sx} x^{\rho-1} S_N^M[(-z_{r+1})x] H\begin{bmatrix} z_1 x \\ \vdots \\ z_r x \end{bmatrix} dx \bar{f}(s) = \bar{g}(s). \tag{2.10}$$

Now expressing the $S_N^M[(-z_{r+1})x]$ and $H\begin{bmatrix} z_1 x \\ \vdots \\ z_r x \end{bmatrix}$ nvolved in (2.10) in series using (1.1) and (1.3), changing the order of series and integration and evaluating the x-integral. We obtain

$$[\sum_{k_1+\cdots+k_{r+1}=B}^{\infty} \Delta(k_1,\ldots,k_{r+1}) \frac{(-z_1)^{k_1}}{k_1!} \cdots \frac{(-z_{r+1})^{k_{r+1}}}{k_{r+1}!} \times s^{-\rho-(k_1+\cdots+k_{r+1})}] \bar{f}(s) = \bar{g}(s) \tag{2.11}$$

where $\Delta(k_1,\ldots,k_{r+1})$ is defined by (2.6). Now making use of the known formula [5, p. 67. eq. (2.3)]. we easily obtain from (2.11)

$$[\sum_{B=0}^{\infty} \lambda_B s^{-B}] s^{-\rho} \bar{f}(s) = \bar{g}(s) \tag{2.12}$$

where λ_B is defined by (2.5).

Again. (2.12) is equivalent to

$$\bar{f}(s) = s^{-\rho}[\sum_{B=0}^{\infty} \lambda_B s^{-B}]^{-1} \bar{g}(s) \tag{2.13}$$

If μ denotes the least B for which $\lambda_B \neq 0$ the series given by (2.13) can be reciprocated. Writing

$$[\sum_{B=0}^{\infty} \lambda_{B+\mu} s^{-B}]^{-1} = \sum_{j=0}^{\infty} E_j s^{-j} \tag{2.14}$$

eq. (2.13) takes the following form:

$$\bar{f}(s) = s^{\rho-l+u} \sum_{j=0}^{\infty} E_j s^{-j} [s^l \bar{g}(s)] \tag{2.15}$$

(2.15) can be written as

$$L\{f(x); s\} = L\{\sum_{j=0}^{\infty} \frac{(E_j x^{j+l-\rho-\mu-1})}{\Gamma(j+l-\rho-\mu)}; s\} L\{g^{(l)}(x); s\} \tag{2.16}$$

[on using (1.6)].

Now using the convolution theorem in the RHS of (2.16) we get

$$L\{f(x); s\} = L\{\int_0^x \sum_{j=0}^{\infty} \frac{E_j (x-u)^{j+l-\rho-\mu-1}}{\Gamma(j+l-\rho-\mu)} g^{(l)}(u) du; s\} \tag{2.17}$$

Finally, on taking the inverse of the Laplace transform of both sides of (2.17) we arrive at the desired result (2.2).

Special Cases:

In the main result if we take N =0 (the polynomial S_0^M will reduce to Ao,o Which can be taken to be unity without loss of generality), we arrive at a result given by Srivastava et al [5, p. 64, eq. (1.1)].

Again, if we put r = 1, p = q = 0, z_2 = -1 in the main result, and further reduce the Fox's H-function obtained to exp $(-z_1)$[9. p. 18, eq. (2.6.2)] and let $z_1 \to 0$,the Fox's H-function reduces to unity and we arrive at a result which in essence is the same as that given by Rashmi Jain [2, pp. 102-103, eqs (3.5). (3.6)].

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