# CONCENTRATION PROBABILITY CRITERION AND GENERALIZED CLASS OF ESTIMATORS IN RESTRICTED REGRESSION MODEL

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**ABSTRACT**

Concentration probability criterion and generalized classes of estimators in restricted regression model, to study two generalized families of estimators from the literature in linear regression model under exact linear constraints of parameter vector when the criterion of choice of estimators is taken to be concentration probability of estimator around the true parameter. Following small sigma asymptotic approach, the sampling distributions and the concentration probabilities of the two generalized families of estimators are derived and the efficiency is discussed with respect to the criterion of concentration probability.

## 2. THE MODEL AND THE ESTIMATORS

Consider the classical linear regression model

$$y = x\beta + u$$

Where y is a Tx1 vector observation on the variable to tbe explained, X is a Txp matrix of observations on p explanatory variables, $\beta$ is a px1 vector of unknown regression coefficients and u is a Tx1 vector of disturbances following normal distributions with mean vector zero and dispersion matrix $\sigma^2 I_T, \sigma^2$ being the unknown variance of disturbances. Let the available apriori information on the coefficient vector be in the from of linear constraints, given by

$$q = Q\beta$$

Where q is the (Jx1) (j<P) known vector and Q is a known (Jxp) full row rank matrix.

For the model (2.1), we known that the ordinary least squares (OLS) estimator

$$b = (x'x)^{-1}X'y$$

Is the best linear unbiased estimator of $\beta$ and dispersion matrix $\sigma^2(x'x)^{-1}$

Incorporation of linear constraints in the model, leads to the restricted least squares estimator given by

$$b_R = b + (x'x)^{-1}Q'[Q(x'x)^{-1}Q']^{-1}(q - Qb)$$

Which is unbiased and is distributed normally with mean vector β and variance covariance matrix

$$E(b_R-\beta)(b_R-\beta)' = \sigma^2\Omega$$

Where $\Omega = (x'x)^{-1} - (x'x)^{-1}Q'[Q(x'x)^{-1}Q']^{-1}(x'x)^{-1}$

Satisfying the apriopri restriction (2.2), Srivastava and Chandra (1991) considered the following two families $b_{RS}$ *and* $b_{SR}$ of estimators

$$b_{RS} = b_R - \frac{k(y-xb)'(y-xb)}{b'cb}\Omega x'xb$$

$$= b_R - k\, z\, \Omega\, x'x\, b \ (2.7)$$

*and*

$$b_{SR} = \left[I_p - \frac{K(y-xb_R)'(y-xb_R)}{b_R C b_R}\right] b_R$$

$$= \left[I_p - k\, z^*\, x'x\right] b_R$$

$$where\ z = \frac{(y-xb)'(y-xb)}{b'Cb}, \qquad z^* = \frac{(y-xb_R)'(y-xb_R)}{b_R' C b_R}$$

$k$ *is a* characterizing scalar greater than zero and C is a characterizing positive definite symmetric matrix.

Following small disturbance asymptotic theory and taking general quadratic loss function, Srivastava and Chandra(1991) derived the approximate risk up to the order $O(\sigma^4)$ for the families $b_{RS}$ *and* $b_{SR}$ of estimators and found some dominance conditions for their superiority.

Considering the concentration probabilities of the estimators $b_{RS}$ *and* $b_{SR}$ around β, Shukla (1993) examined their concentration optimality.

Singh (1994) defined two more general families $b_g$ *and* $b_h$ of estimators as

$$b_g = b_R + g(z)\, \Omega\, x'x\, b$$

And

$$b_h = \left[I_p + h(z^*)\, \Omega\ x'x\, \right] b_R$$

Where g(z) and h(z*) satisfying the validity condition of Taylor's (Maclaurian's ) series expansion with appropriate finite expectation and having first two derivatives bounded, are functions are functions of z and z* respectively such that $g(z=0)=0$, $g(z) = O(\sigma^2)$, $h(z^*=0) = 0$, $h(z^*) = O(\sigma^2)$ and z , z* have at least $k(\geq 4)$ finite moments, following small σ asymptotic approach, Singh derived the risk function of $b_g$ *and* $b_h$ with respect to a general quadratic loss functions and studied their properties. Here, we analyze the properties of the two generalized families of estimators $b_g$ *and* $b_h$ by Singh from the criterion of concentration probability around the true unknown parameter and compare them with the existing estimators in search of better ones.

## 3. CONCENTRATION PROBABILITIES OF THE ESTIMATORS $b_g$ AND $b_h$

We first derive the small sigma asymptotic expression for the sampling distribution of the estimators $b_g$ *and* $b_h$.

Rewriting the model (2.1) as

$y = \beta + \sigma \text{v}$

Where v follows multivariate normal distribution with mean we now define a vector

$$r_g = \frac{1}{\sigma}\Omega^{-\frac{1}{2}}(b_g - \beta)$$

$$= A_0 + \sigma A_1 + \sigma^2 A_2 + \sigma^3 A_3 + \sigma^4 A_4 \qquad (3.7)$$

*the characteristic function of* $r_g$ up to terms of order $O(\sigma^3)$ is given by

$$\phi_g(h) = E(e^{ih'r_g})$$

$$= E\left(e^{ih'A_0}\right) e^{\{\sigma ih'A_1 + \sigma^2 ih'A_2 + \sigma^3 ih'A_3 + O(\sigma^4)\}}$$

$$= E\left(e^{ih'A_0}\right)\left[1 + +\sigma(ih'A_1) + \sigma^2\left\{\left(ih'A_2 + \frac{1}{2}(ih'A_1)^2\right)\right\} + \sigma^3\{(ih'A_3 + (ih'A_1)(ih'A_2)\} + \frac{1}{6}(ih'A_1)^3\right]$$

We have the characteristic function

$$\phi_g(h) = [1 + \sigma\phi_1 + \sigma^2\phi_2 + \sigma^3\phi_3]\, e^{-\frac{1}{2}h'h}$$

By inversion theorem, the joint probability density function of the elements of $r_g$ is given by

$$g(r_g) = \frac{1}{(2\pi)^P}\int_{-\infty}^{\infty} \ldots\ldots \int_{-\infty}^{\infty} e^{-ih'r_g}\,\phi_g(h)\, dh \qquad (3.11)$$

Substituting $\phi_g(h)$ from (3.10) in (3.11) and for a fixed vector a and a fixed matrix A, utilizing the following result;

we obtain the joint probability density function of $r_g$ to order $O(\sigma^3)$ to be

$$g(r_g) = (1 + \sigma\varsigma_1 + \sigma^2\varsigma_2 + \sigma^3\varsigma_3)\,\xi(r_g)$$

For $\bar{m} = (\bar{m}_1, \bar{m}_2 \ldots\ldots \bar{m}_p)$, when $\bar{m}_j$, j=1,2 ....P are arbitrarily chosen positive constants, the concentration probability associated with the estimator $b_g$ around the parameter vector $\beta$ in the region bounded by the constants $\bar{m}_1, \bar{m}_2 \ldots\ldots \bar{m}_p$ in the p - dimensional ecludian space is given by

$$CP(b_g) = P\,(\,|b_g - \beta| < \,-)$$

$$= P\{|r_{gj}| < m_j; j = 1,2,3, \ldots\ldots, p\}$$

where $r_{gj}$ is the $J^{th}$ element of the vector $r_g$ and the constants $m_j's(j = 1,2, \ldots, p)$ are the elements of the vector $m = \frac{1}{\sigma}\Omega^{-\frac{1}{2}}\overline{m}$

For fixed vector a and fixed matrix A, and for the region bounded by the column vector m in the p- dimensional Ecludian space, we have

Nothing the concentration probability cf estimator $b_g$ around $\beta$ for the region bounded by the constants $\bar{m}_1$, $\bar{m}_2$ …… $\bar{m}_p$ in the p- dimensional Ecludian space to be

$$CP(b_g) == \int_{-m_p}^{m_p} \ldots\ldots \int_{-m_1}^{m_1} g(r_g)\ dr_{g1} \ldots\ldots dr_{gp}$$

and using the results of(3.12) and (3.13) in (3.14), we have

$$CP(b_g) = \left[1 - \frac{ng'(0)\sigma}{\theta}\left\{tr\ A\ E + \frac{(n+2)g'(0)}{2\theta}\ (\alpha_1 E \alpha_1)\right\}\right] \phi($$

where E = diag. $(e_1, e_2, \ldots, e_p)$

For g'(0) = K ( a characterizing scalar greater than 0) and Q → 0, $b_g$ reduces to stein – rule estimator

$$b_s = \left[1 - k\frac{(y - xb)'(y - xb)}{b'Cb}\right] b,$$

for K=0 and Q≠0, $b_g$ reduces to the restricted regression estimator $b_R$ and for K = 0 and Q → 0, $b_g$ reduces to the ordinary least square estimator b so that the concentration probabilities of the stein – rule estimator $b_s$, the restricted regression estimator $b_r$ and the ordinary least square estimator b around the parameter $\beta$ are given by

$$CP(b_s) = \left[1 + \frac{nk\rho^2}{\theta}\left\{tr\ \tilde{A}\ E - \frac{(n+2)k}{2\theta}.(\alpha_1^*\ E\alpha_1^*)\right\}\right]$$

$$CP(b_r) = \phi(\mathrm{m}) \quad (3.18)$$

$$CP(b) = \phi(\mathrm{m}^*) \quad (3.19)$$

$$\phi(\mathrm{m}^*) = \int_{-m}^{m} \ldots\ldots \int_{-m_1}^{m_1} \xi(r_g)\ dr_{g1}\ \ldots\ldots dr_{gp}$$

On the same lines as for $b_g$ the concentration probability of the estimator $b_h$ around $\beta$ is given by

$$CP(b_h) = \left[1 - \frac{(n+J)h'(0)\rho^2}{\theta}\left\{tr\ A\ E + \frac{(n+J+2)h'(0)}{2\theta}.(\alpha_1 E\alpha_1)\right\}\right] \phi(\mathrm{m}), (3.20)$$

Where h' (0) is the first derivative of h(z* ) with respect to z* at z*=0, E= diag. $(e_1, e_2, \ldots, e_p)$

## 4 COMPRISON OF CONCENTRATION PROBILITIES

To compare the performance of the generalized estimator $b_g$ with the restricted regression estimator $b_R$ on the criterion of concentration probability, we have

$$CP(b_g) - CP(b_R) = \left[\frac{-ng'(0)\sigma^2}{\theta}\left\{tr\ A\ E + \frac{(n+2)g'(0)}{2\theta}\ .(\alpha_1' E\alpha_1)\right\}\right] \phi(\mathrm{m})$$

We observe that the estimator $b_g$ is superior to the restricted regression estimator $b_R$ based on the criterion of concentration probability to order $0(\sigma^3)$,if

$$0 < -g'(0) < \frac{2tr\ A\ E\ \beta' C\beta}{(n+2)\alpha_1' E\alpha_1},$$

$$0 < -g'(0) < \frac{2}{n+2}\frac{\sum_j^p e_j^* - 2e_P^*}{\bar{C}\bar{h}\left[E\Omega^{\frac{1}{2}}(x'x)C^{-1}(x'x)G\Omega^{\frac{1}{2}}\right]}$$

For C = (X'X) or $\Omega^{-1}$ , the superiority condition of the estimator $b_g$ over $b_R$ turns out to be

$$0 < -g'(0) < \frac{2}{n+2}\frac{\left(\sum_{j=1}^P e_j^* - 2e_P^*\right)}{e_P^*}$$

In particular when all the elements of constant vector m are equal, that is,

*when* $m_j = m_0$ *and*

$$e_j = \frac{m_0 e^{-\frac{1}{2}m_0^2}}{\int_0^{m_0} e^{-\frac{1}{2}r_{gj}^2 dr_{gj}}} \quad ; \qquad j = 1,2,3,4 \ldots \ldots \ldots P,$$

The concentration dominance condition of the estimator $b_g$ over $b_R$ becomes

$$0 < -g'(0) < \frac{2(p-2)}{n+2}$$

To compare the performance of the generalized estimator $b_h$ with restricted regression estimator $b_r$ on the criterion of concentration probability, we have

$$CP(b_h) - CP(b_R) = -\frac{(n+J)}{\theta}h'(0)\sigma^2\left\{tr\ AE - \frac{(n+J+2)h'(0)}{2\theta}(\alpha_1' E\alpha_1)\right\}\phi(\mathrm{m})$$

The estimator bh is superior to the restricted regression estimator bR base on the criterion of concentration probability to order $O(\sigma^3)$, if

$$0 < \ -h'(0) < 2tr\ AE\frac{\beta' C\beta}{(n+J+2)(\alpha_1' E\alpha_1)}$$

Which holds true at least as long as

$$0 < \ -h'(0) < \frac{2\left(\sum_{j=1}^p e_j^* - 2e_p^*\right)}{(n+J+2)\overline{Ch}\left\{E\Omega^{\frac{1}{2}}(x'x)C^{-1}(x'x)\ \Omega^{\frac{1}{2}}\right\}}$$

For $C = (X'X)\ or\ \Omega^{-1}$ ,the superiority condition of the estimator $b_h$ over $b_R$ becomes

$$0 < [-h'(0)] < \frac{2}{(n+J+2)}\frac{\left(\sum_{j=1}^p e_j^* - 2e_p^*\right)}{e_p^*}$$

In particular, when all the elements of the constant vector $m$ are equal, $m_j = m_0$ and

$$e_j = \frac{m_0\left(e^{-\frac{1}{2}m_0^2}\right)}{\left(\int_0^{m_0} e^{-\frac{1}{2}r_{gj}^2 dr_{gj}}\right)}; j = 1,2,3,, \ldots\ldots, p,$$

The concentration dominance condition of the estimator $b_h over\ b_R$becomes

$$0 < [-h'(0)] < \frac{2(p-2)}{(n+J+2)}$$

## 5. CONCLUDING REMARK

(a) All the result of Shukla (1993) may be easily seen to be special cases of this general study based on the concentration probability around true parameter. In particular, it may be easily seen that the value of $g'(0)\ is\ -k$ for the estimator $b_{RS}$ so that by substituting this value $g'(0) = -k$ in general efficiency condition based on the criterion of concentration probability around the true coefficient vector $\beta$, we get the same condition for $b_{RS}$ to be better then $b_R$ as obtained by Shukla (1993).

(b) For k and $k_1$ being the characterizing scalars, the estimator

$$b_{g_1} = b_R - k[(1+z)^{k_1} - 1]\, \Omega x'\, x\, b$$

Belonging to the generalized class $b_g$ of estimators, has the value

$$g'(0) = -kk_1$$

Which, when substituted in the general efficiency condition gives the efficiency condition

$$0 < kk_1 < \frac{2(p-2)}{(n+2)}$$

For $b_{g1} to$ be better than $b_R$ based on the concentration probability criterion. It is to be noted that, for $k_1 = 1$ , the efficiency condition reduces to the condition $0 < k_1 < 2\frac{(p-2)}{(n+2)}$

For $b_{RS}\ to$ be better then $b_R$ as obtained by Shukla (1993). Further, for $0 < k_1 < 1$ , the range of the condition for the $b_{g1}$ to be better then $b_R$ is wider than that of the condition for $b_{RS}$ to be better than

$b_r$, hence in the extended range of the efficiency condition over the efficiency condition the estimator $b_{g1}$ is better than both the estimators $b_{RS}$ and $b_R$ in the sense of having more concentration probability around the true parameter $\beta$.

(c ) Considering the estimator

$$b_{h_1} = b_R - k[(1+z^*)^{k_1} - 1\,]\, \Omega\, x'x\, b_R$$

Belonging to the generalized class $b_h$ of estimators, we have the value of

$$h'(0) = -k\, k_1$$

And similar results as for the comparison of $b_{g_1}\ and\ b_R$ , hold while comparing the estimator $b_{h_1}\ with\ b_{SR}$.

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