α - Graceful Labeling for a Binary Tree and Graceful Labeling for a Regular Tree

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Key words : Graceful graph, α-graceful graph, One point union of two graphs, Banana tree, Regular binary tree

AMS **subject classification** (2010) **:** 05C78.

Abstract

Labeled graph is the topics of current interest and here we have discussed α-graceful labeling for the regular binary tree. We have also discussed graceful labeling for banana tree, symmetric tree and regular tree.

1 Introduction

In 1996 Rosa defined graceful labeling of a simple graph G and α-labeling (here we call α-graceful labeling) for a graph. Banana tree $B(n, k)$, to be the tree obtained by joining one leaf of each n copies $K_{1,k-1}$ ($(k-1)$ - star) i.e. it is one point union of n copies of star graph $K_{1,k-1}$. A symmetric tree $T_{k+1}(d)$, to be a tree with diameter d, in which all vertices other than leaves and root have the same degree $k+1$ and all leaves have same eccentricity, where root is the center for $T_{k+1}(d)$, with degree k and eccentricity $\frac{d}{2}$. Here d is the diameter for $T_{k+1}(d)$. $d_G(v)$ is denoted for the degree of vertex v in G.

In this paper a graph G we mean a simple, finite and undirected graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. We follow Harary[1] for basic notation and terminology of graphs.

2 Main Results

Theorem 2.1

Let T be a graceful tree. Let f be a graceful labeling for T and there is $v \in V(T)$, $d_T(v) = 1$ and $f(v) = 0$. Then one point union of two copies of T at v is α - graceful tree.

Proof: Let $p = |V(T)|$ then $q = |E(T)| = p - 1$. Let $V(T^{(1)}) = \{v_0 = v, v_1, v_2, \ldots, v_q\}$ be vertices of first copy $T^{(1)}$ of T. Let f be a graceful labeling for $T = T^{(1)}$ such that $f(v_0) = 0$. Let $T^{(2)}$ be another copy of T and $V(T^{(2)}) = \{u_0, u_1, u_2, \ldots, u_q\}$. Let G be a graph (tree) obtained by merging $u_0 = v_0 = v, v_1, v_2, \ldots, v_q, u_1, u_2, \ldots, u_q\}$ and $|E(G)| = 2q$. Since $T = T^{(1)}$ is bipartite graph, for each $e = (u, w) \in E(T)$, there is a partition $V_1 \cup V_2$ of $V(T)$ such that $u \in V_1$ and $w \in V_2$. Take $v \in V_1$.

Define $g : V(G) \to \{0, 1, 2, \ldots, 2q\}$ by $g/V_1^{(1)} = f/V_1^{(1)}$, $g/V_2^{(1)} = f/V_2^{(1)} + q$, $g/V_1^{(2)} - \{v\} = f/V_1^{(1)} - \{v\} + q$ and $g/V_2 = f/V_2^{(1)}$ where $V_i^{(i)} \cup V_2^{(i)}$ is the vertex partition of $V(T^{(i)})$, $i = 1, 2$. First we shall prove here g is a bijection.

Let $w_1, w_2 \in V(G)$ be such that $g(w_1) = g(w_2)$ and $w_1 \neq w_2$.

$\Rightarrow$ $f(w_1) = f(w_2)$ and $w_1 \in V_1$, $w_2 \in V_2$ which is impossible as f is one-one.

Since, $|V(G)| = 2q + 1$, $g : V(G) \to \{0, 1, ..., 2q\}$ must be a bijection. The induced edge labeling $g^* : E(G) \to \{1, 2, \ldots, 2q\}$ defined by $g^*(e = (w_1, w_2)) = |g(w_1) - g(w_2)|$. Now we shall prove g^* is bijective map.

Let $e_1 = (w_1, w_2)$, $e_2 = (w_3, w_4) \in E(G)$ such that $g^*(e_1) = g^*(e_2)$ where $w_1, w_3 \in V_1$.

$\Rightarrow |g(w_1) - g(w_2)| = |g(w_3) - g(w_4)|$

$\Rightarrow | \pm q + (f(w_1) - f(w_2))| = | \pm q + (f(w_3) - f(w_4))|$

$\Rightarrow q \pm |(f(w_1) - f(w_2))| = q \pm |(f(w_3) - f(w_4))|$

$\Rightarrow$ $f(w_1) - f(w_2)$, $f(w_3) - f(w_4)$ either both are positive or both are negative.

$\Rightarrow |f(w_1) - f(w_2)| = |f(w_3) - f(w_4)|$

$\Rightarrow f^*(e_1) = f^*(e_2)$

$\Rightarrow e_1 = e_2$, as f^* is a bijection.

Now for any $e = (w_5, w_6) \in E(G)$,

$g^*(e) = |g(w_5) - g(w_6)|$

$=$ either $q + |f(w_5) - f(w_6)|$ or $q - |f(w_5) - f(w_6)|$

$=$ either $q - f^*(e)$ or $q + f^*(e)$

$\Rightarrow g^*(E(G)) = \{1, 2, \ldots, q, q + 1, q + 2, \ldots, 2q\}$

Thus, above labeling pattern g gives rise to a graceful labeling to the graph (tree) G.

Let $w_7, w_8 \in V(T)$ be such that $f(w_7) = q$ and $f(w_8) = 1$. Since $f(v) = 0$ and $d_T(v) = 1$, v is adjacent only with the vertex w_7 in T. To produce the edge label $q - 1$ in T, w_8 should be adjacent with w_7.

Now $g^*(w_7, w_8) = |g(w_7) - g(w_8)|$

$= q + |f(w_7) - f(w_8)|$ or $q - |f(w_7) - f(w_8)|$

$= q - (f(w_7) - f(w_8))$ in second copy $T^{(2)}$

$= 1$ in second copy $T^{(2)}$

Take $k = g(w_7) = q$. It is observed that for any $e = (w_9, w_{10}) \in E(G)$, $min\{g(w_9), g(w_{10})\} \leq k = q < max\{g(w_9), g(w_{10})\}$ and so, g is an α-graceful labeling for G.

2.2 Regular binary tree :

Regular binary tree BT_n, where $n = 1 + 2 + \ldots + 2^m$, for some $m \in N$ i.e. $BT_1 = K_1$, $BT_3 = P_3$ and BT_7 is obtained by taking one point union of two copies of $K_{1,3}$ as shown in figure - 1.

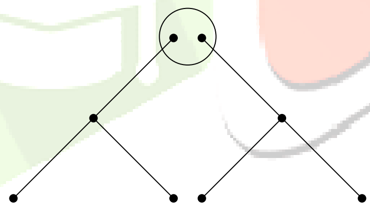

Figure - 1

one point union of two copies of $K_{1,3}$.

Next step, add one pendent vertex at the common vertex of $2K_{1,3}$ in BT_7 and to obtain BT_{15}, take one point union of two copies of above said tree by murging the added pendent vertex as shown in figure - 2.

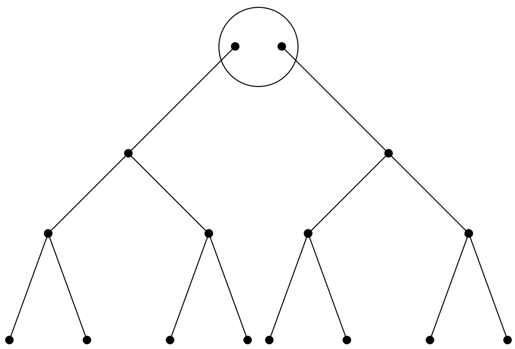

Figure - 2

one point union of two copies of graph obtained by adding a pendent vertex at the root of $BT7$.

Continue this way, add one pendent vertex at the root of BT_{2^m-1} and to obtain $BT_{2^{m+1}-1}$ take one point union of two copies of BT_{2^m-1} with pendent vertex by murging the added pendent vertex as shown in figure - 3.

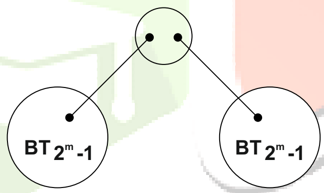

Figure - 3

one point union of two copies of $BT_{2^{m+1}-1}$ after adding pendent vertex at the root.

Thus, $BT_{2^{m+1}-1}$ is the symmetric tree $T_3(2m)$.

2.3 Algorithm to obtain α-graceful labeling BT_{2^m-1} :

Obviously, following graceful labeling (given in figure - 4) for $K_{1,3}$ is an α-graceful labeling, where $k = 2$.

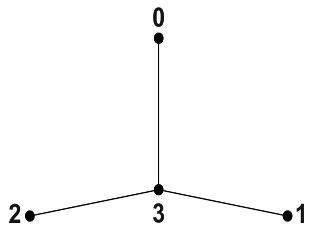

Figure - 4

α-graceful labeling for $K_{1,3}$

Using this according to **Theorem. 2.1** obtain α-graceful labeling for BT_7 as shown in figure - 5.

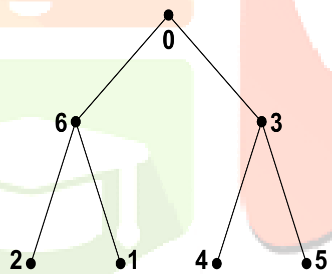

Figure - 5

α-graceful labeling for BT_7

Add one pendent vertex with vertex label 7, which gives an α-graceful labeling and take its complement α-graceful labeling by subtracting each vertex label from 7 and according to **Theorem 2.1** obtain α-graceful labeling for BT_{15} as shown in figure - 6.

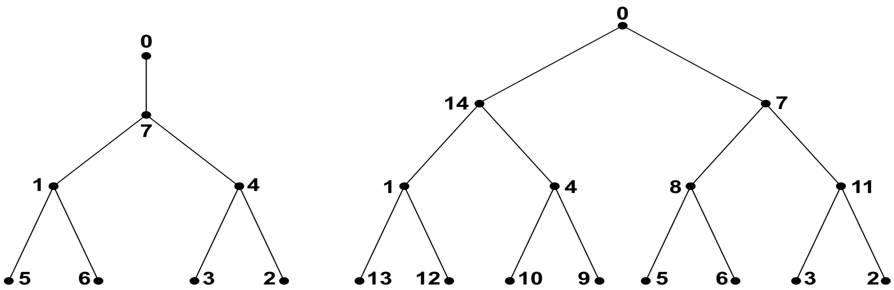

Figure - 6

α-graceful labeling for the graph obtained from BT_7 by adding one pendent vertex and BT_{15}

Theorem 2.4

Let T be graceful tree. Let f be a graceful labeling for T and there is $v \in V(T)$, $d_T(v) = 1$ and $f(v) = 0$. Then one point union of three copies of T at v is graceful.

Proof: Let $p = |V(T)|$ then $q = |E(T)| = p - 1$. Let $V(T^{(1)}) = \{v_0 = v, v_1, v_2, \ldots, v_q\}$ be vertices of first copy $T^{(1)}$ of T. Let f be an arbitrary graceful labeling for $T = T^{(1)}$ such that $f(v_0) = 0$. Let $T^{(2)}$, $T^{(3)}$ be another copies of T and $V(T^{(2)}) = \{u_0, u_1, u_2, \ldots, u_q\}$, $V(T^{(3)}) = \{w_0, w_1, w_2, \ldots, w_q\}$. Let G be a graph (tree) obtained by merging $u_0, v_0, w_0\}$ (one point union of $T^{(i)}, i = 1, 2, 3)$.

It is obvious that $V(G) = \{v, v_1, \ldots, v_q, u_1, \ldots, u_q, w_1, \ldots, w_q\}$ and $|E(G)| = 3q$. Since $T = T^{(1)}$ is bipartite graph, for each $e = (u, w) \in E(T)$, there is a partition $V_1 \cup V_2$ of $V(T)$ such that $u \in V_1$ and $w \in V_2$. Take $v \in V_1$.

Define $g : V(G) \to \{0, 1, 2, \ldots, 3q\}$ by $g/V_1^{(1)} = f/V_1^{(1)}$, $g/V_2^{(1)} = f/V_2^{(1)} + 2q$, $g/V_1^{(2)} - \{v\} = 2q + f/V_1^{(1)} - \{v\}$ and $g/V_2^{(2)} = f/V_2^{(1)}$ and $g/V(T^{(3)}) - \{v\} = q + f/V(T^{(3)}) - \{v\}$ where $V_1^{(i)} \cup V_2^{(i)}$ is the vertex partition of $V(T^{(i)})$, $i = 1, 2$. First we shall prove here g is a bijective map.

Let $s_1, s_2 \in V(G)$ be such that $g(s_1) = g(s_2)$ and $s_1 \neq s_2$ if possible.

$\Rightarrow f(s_1) = f(s_2)$ which is not possible as f is one-one.

Thus, $g : V(G) \to \{0, 1, ..., 3q\}$ is a bijection, as g is one-one and $|V(G)| = 3q + 1$.

The induced edge labeling $g^* : E(G) \to \{1, 2, \ldots, 3q\}$ defined by $g^*(e = (s_3, s_4)) = |g(s_3) - g(s_4)|$. We have to prove g^* is also a bijective map. It is enough to prove g^* is one - one.

Let $e_1 = (s_5, s_6)$, $e_2 = (s_7, s_8) \in E(G)$ and $g^*(e_1) = g^*(e_2)$.

$\Rightarrow |g(s_5) - g(s_6)| = |g(s_7) - g(s_8)|$

$\Rightarrow |f(s_5) - f(s_6)| = |f(s_7) - f(s_8)|$, by definition of g

$\Rightarrow f^*(e_1) = f^*(e_2)$

$\Rightarrow e_1 = e_2$, as f^* is a bijection.

Now for any $e = (s_9, s_{10}) \in E(G)$,

$g^*(e) = |g(s_9) - g(s_{10})|$

$=$ either $2q + |f(s_9) - f(s_{10})|$ or $2q - |f(s_9) - f(s_{10})|$ or $|f(s_9) - f(s_{10})|$

$=$ either $2q + f^*(e)$ or $2q - f^*(e)$ or $f^*(e)$

$\Rightarrow g^*(E(G)) = \{1, 2, \ldots, 3q\}$

Thus, above labeling pattern give rise a graceful labeling g to the given graph (tree) G.

Theorem 2.5

Let T be graceful tree. Let f be a graceful labeling for T and there is $v \in V(T)$, $d_T(v) = 1$ and $f(v) = 0$. Then one point union of l copies of T at v is also a graceful tree.

Proof: Let $V(T) = \{v_0 = v, v_1, \ldots, v_q\}$. Let f be a graceful labeling for $T = T^{(1)}$ with $f(v_0) = 0$. Let $T^{(2)}, T^{(3)}, \ldots, T^{(l)}$ be another copies of T and G be a tree obtained by merging vertex v of each copies $T^{(1)}, T^{(2)}, \ldots, T^{(l)}$.

It is obvious that $V(G) = lq + 1$ and $|E(G)| = lq$. Since, $T^{(1)}$ is bipartite graph, for each $e = (u, w) \in E(T)$, there is a vertex partition $V_1 \cup V_2$ of $V(T)$ such that $u \in V_1$ and $w \in V_2$. Take $v \in V_1$.

To define $g : V(G) \to \{0, 1, 2, \ldots, lq\}$ we consider following two cases.

Case - I : l is even.
$g/V_1^{(1)} = f/V_1^{(1)}$, $g/V_2^{(1)} = f/V_2^{(1)} + (l-1)q$, $g/V_1^{(2)} - \{v\} = (l-1)q + f/V_1^{(1)} - \{v\}$, $g/V_2^{(2)} = f/V_2^{(1)}, g/V_1^{(i)} - \{v\} = \left(\frac{i-1}{2}\right) q + f/V_1^{(1)} - \{v\}$, $g/V_2^{(i)} = \left(l - \frac{i+1}{2}\right) q + f/V_2^{(i)}$, $g/V_1^{(i+1)} - \{v\} = \left(l - \frac{i+1}{2}\right) q + f/V_1^{(1)} - \{v\}$, $g/V_2^{(i+1)} = \left(\frac{i-1}{2}\right) q + f/V_2^{(1)}$, $\forall i = 3, 5, \ldots, l-1$

Case - II : l is odd.
$g/V_1^{(1)} = f/V_1^{(1)}$, $g/V_2^{(1)} = (l-1)q + f/V_2^{(1)}$, $g/V_1^{(2)} - \{v\} = (l-1)q + f/V_1^{(1)} - \{v\}$, $g/V_2^{(2)} = f/V_2^{(1)}, g/V_1^{(i)} - \{v\} = \left(\frac{i-1}{2}\right) q + f/V_1^{(1)} - \{v\}$, $g/V_2^{(i)} = \left(l - \frac{i+1}{2}\right) q + f/V_2^{(i)}$, $g/V_1^{(i+1)} - \{v\} = \left(l - \frac{i+1}{2}\right) q + f/V_1^{(1)} - \{v\}$, $g/V_2^{(i+1)} = \left(\frac{i-1}{2}\right) q + f/V_2^{(1)}$, $\forall i = 3, 5, \ldots, l-2$ and $g/V(T^{(l)}) = f/V(T^{(l)}) + \left(\frac{l-1}{2}\right) q$.

Where $V_1^{(i)} \cup V_2^{(i)}$ is the vertex partition of $V(T^{(i)})$, $i = 1, 2, \ldots, l-1$.

Thus, above labeling pattern give rise a graceful labeling g to the given graph (tree) G and so, G is a graceful graph.

Corollary 2.6

Every banana tree $B(n, k)$ is graceful.

Since $B(n, k)$ is one point union of n copies of the star graph $K_{1,k-1}$ and $K_{1,k-1}$ has required graceful labeling, by **Theorem 2.5**, we can obtain a graceful labeling for $B(n, k)$

Corollary 2.7

Symmetric tree T_{n+1} is a graceful graph.

$T_{(n+1)}(2) = K_{1,n}$, $T_{n+1}(4) = B(n, n+1)$, $T_{n+1}(6)$ is one point union of n copies of a graph obtained by adding a pendent vertex to $T_{n+1}(4)$ at the root.

Similarly, $T_{n+1}(d)$ is one point union of n copies of a graph obtained by adding a pendent vertex to $T_{n+1}(d-2)$ at the root.

To get graceful labeling for $T_{n+1}(d)$, use **Algorithm** 2.3 **and Theorem** 2.5 recursively, as $T_{n+1}(2) = K_{1,n}$, $T_{n+1}(4) = B(n, n+1)$ both are graceful trees.

2.8 Graceful Labeling and Regular Tree :

Regular tree is $R(t_1, t_2, \ldots, t_l)$, where $t_1, t_2, \ldots, t_l \in N$ and we define it as follows.

Take t_3 copies of a banana tree $B(n, k)$ and add one pendent edge to each copy of $B(n, k)$ at the root. Now take one point union of t_3 copies of the graph obtain from $B(n, k)$ at the pendent vertex, as shown in figure-7.

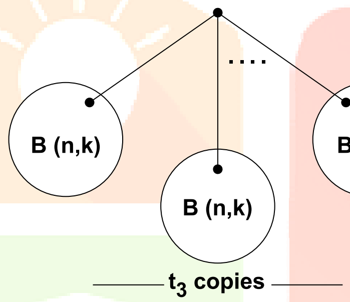

Figure - 7

Such regular graph we denote by $R(k, n, t_3)$. By taking $n = t_2$ and $k = t_1$, we rewrite it by $R(t_1, t_2, t_3)$. Next take t_4 copies of $R(t_1, t_2, t_3)$ and add one pendent edge to each copy at root. Then take one point union of t_4 trees at the added pendent vertex to get the regular tree $R(t_1, t_2, t_3, t_4)$.

Continuing in this way, we get $R(t_1, t_2, t_3, \ldots, t_{(l-1)})$ with a pendent edge at the root. It is obvious that $BT_7 = B(2, 3)$, $BT_{15} = R(3, 2, 2)$ and $BT_{2^{m+1}} - 1 = R(3, 2, \ldots, 2(m\text{-times}))$

To obtain graceful labeling for $R(t_1, t_2, t_3, \ldots, t_l)$ to get recursively way graceful labeling for $R(t_1, t_2, t_3, \ldots, t_{(l-1)})$, $R(t_1, t_2, t_3, \ldots, t_{(l-2)})$, $R(t_1, t_2, t_3)$ and $B(t_2, t_1)$ obtain by **Corollary** 2.6.

Illustration 2.9

$R(4,5,3)$ with its graceful labeling,to obtain this a graph obtain by adding a pendent edge to $B(5,4)$ and $B(5,4)$, $K_{1,3}$ with their graceful labeling are shown in figure - $10, 9, 8$.

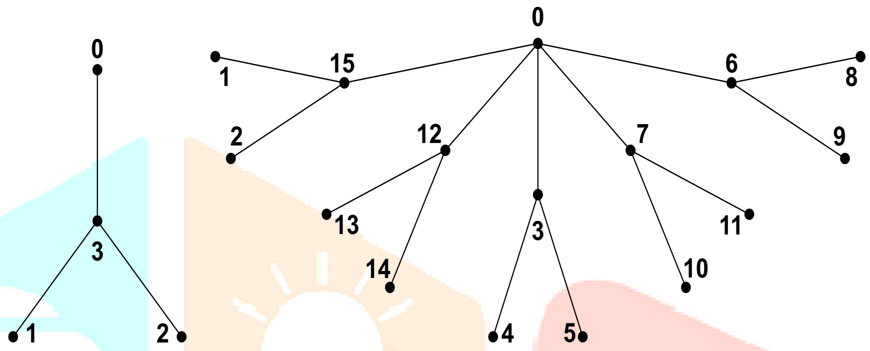

Figure - 8

$B(5,4), K_{1,3}$ and their graceful labeling

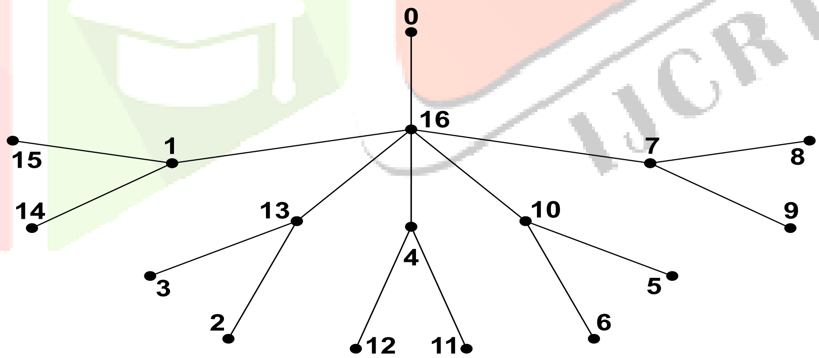

Figure - 9

A graph obtained by adding a pendent edge to $B(5,4)$ and its complement graceful labeling

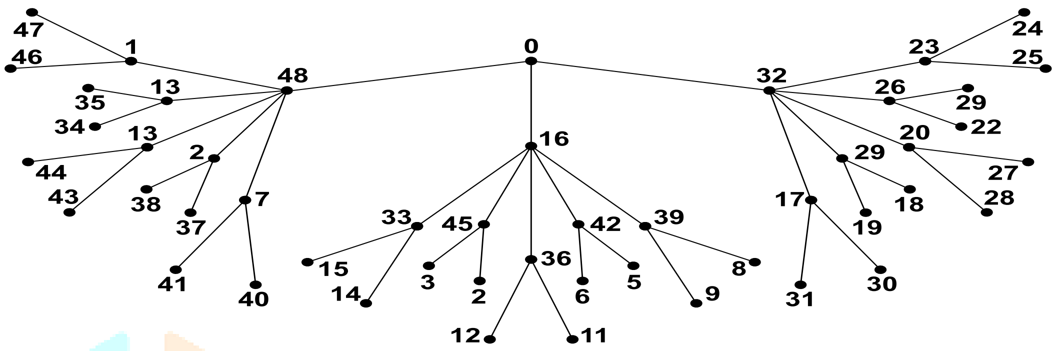

Figure - 10

$R(4,5,3)$ **and its graceful labeling otbained by Theorem** 2.5

References

[1] F. Harary, Graph Theory, *Addition Wesley, Massachusetts*, 1972.

[2] J. A. Gallian, A dynamic survey of graph labeling, *The Electronics J. Combin.*, 19, ♯DS6 (2016).

[3] A. Rosa, On certain valuations of the vertices of a graph, *Thoery of Graphs (Int. Symposium, Rome July 1966), Gordon and Breach, N.Y. and Dunod Paris* (1967), pp. 349-355.