On Almost Complex Manifold in Complex Structure {F}

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Abstract: - **Complex structure {F} is almost complex manifolds is shown. It has been defined and studied by Schouten and Dontzing (1930) introduced the concept of complex structure and a Hermitian metric in a differentiable manifold and called it a complex manifold In this article we discuss the, Almost complex structure $\{F\}$ is not unique and also discuss that the complex structure $\{F\}$ has 2m Eigen values. An almost complex manifold is that it contain a tangent bundle π_m of *dim m* and a tangent bundle $\tilde{\pi}_m$ conjugate to π_m such that $\pi_m \cap \tilde{\pi}_m = \phi$ and they span together a tangent bundle of *dim 2m*.**

Keywords:- **Differentiable manifolds, Complex Manifold, Complex Structure, Nijenhuis Tensor, Contravariant vector, covariant vector, symmetric connection, linear manifold of dim 2m, tangent bundle, linearly independent.**

Introduction: - Let V_n, $n = 2m$ be an even dimensional differentiable manifold of differentiability class C^{r+1} and let there exists a vector valued real linear function *F* of differentiability class C^{r+1} on V_n, satisfying

$$F^2 + I_n = 0 \iff \bar{\bar{X}} + X = 0. \qquad X \in T_p$$

For arbitrary vector field *X*, where $\bar{X} = FX$.

Then V_n is said to be an almost complex manifold and $\{F\}$ is said to give an almost complex structure on V_n .

Example.1: Let us consider V_4 , on which *F* given by,

$$F = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{4\times 4}$$

Therefore,

$$F^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{4\times 4}$$

i.e. $\qquad F^2 + I_4 = 0$.

Example.2: Let $m = R^2$, considered as a manifold with local coordinate the ordinary Cartesian coordinates (x, y). For each $p \in M$ the endomorphism of M_p given by,

$$J_p : a\left(\frac{\partial}{\partial x}\right)_p + b\left(\frac{\partial}{\partial y}\right)_p \longrightarrow -b\left(\frac{\partial}{\partial x}\right)_p + a\left(\frac{\partial}{\partial y}\right)_p$$

Let $\qquad X = a\left(\frac{\partial}{\partial x}\right)_p + b\left(\frac{\partial}{\partial y}\right)_p$

Then $\qquad J_p X = -b\left(\frac{\partial}{\partial x}\right)_p + a\left(\frac{\partial}{\partial y}\right)_p$

$$J_p^2 X = -a\left(\frac{\partial}{\partial x}\right)_p - b\left(\frac{\partial}{\partial y}\right)_p = -X$$

$$\left(J_p^2 + I\right)X = 0$$

Since X is arbitrary vector field, therefore we have

$$\left(J_p^2 + I\right) = 0.$$

Theorem: The $rank(F) = n$.

Proof: Let $\overline{X} = 0 \Rightarrow \overline{\overline{X}} = 0 \Rightarrow X = 0$

Hence $\overline{X} = 0$ has only trivial solution $X = 0$, consequently $rank(F) = n$.

Theorem:- Almost complex structure $\{F\}$ is not unique.

Proof:- Let us define

(1) $$\mu F' = F\mu$$

Where μ is a non-singular tensor of type $(1,1)$ and $\{F\}$ is an almost complex structure,

Post multiplying (1) by F', we get

$$(\mu F')F' = (F\mu)F'$$

$$\mu F'^2 = F(\mu F') = F(F\mu) = F^2\mu = -\mu$$

Therefore $F'^2 + I_n = 0$ since $\mu \neq 0$

i.e. $\{F'\}$ is an almost complex structure.

Nijenhuis Tensor: Nijenhuis with respect to F is a vector valued bilinear function N, given by

$$N(X,Y) = [F,F](X,Y) \overset{def}{=} [\overline{X},\overline{Y}] + \overline{\overline{[X,Y]}} - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]}.$$

In an almost complex manifold,

$$N(X,Y) = [F,F](X,Y) \overset{def}{=} [\overline{X},\overline{Y}] - [X,Y] - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]}$$

Theorem: On an almost complex manifold, we have

(i) $N(\overline{X},\overline{Y}) = -N(X,Y) = -\overline{N(\overline{X},Y)} = -\overline{N(X,\overline{Y})}$

(ii) $\overline{N(\overline{X},\overline{Y})} = -\overline{N(X,Y)} = N(\overline{X},Y) = N(X,\overline{Y})$

Proof: Proof is obvious.

Definition: An almost complex manifold with vanishing Nijenhuis tensor is a complex manifold.

Definition: On an almost complex manifold V_n, a bilinear function A is said to be

Pure if $A(X,Y) + A(\overline{X},\overline{Y}) = 0$

Hybrid if $A(X,Y) = A(\overline{X},\overline{Y})$.

Theorem: F has m Eigen values $+i$ and m Eigen values $-i$.

Proof: Let λ be an Eigen values of F and let P be the corresponding eigen vector. Then

$$FP = \lambda P$$

i.e. $\overline{P} = \lambda P$.

Barring P we get,

$$\overline{\overline{P}} = \lambda^2 P$$

i.e. $\lambda^2 = -1$ (Since $P \neq 0$)

Therefore $\lambda = \pm i$

Thus $+i$ and $-i$ are the Eigen values of *F*. Since $n=2m$, show $+i$ repeated *m* times, $-i$ repeated *m* times.

Theorem: Let $\{F\}$ and $\{F'\}$ be two almost complex structures of the almost complex manifold V_n connected by $\mu F' = F\mu$ then, if *P* is an Eigen vector of F', μP is an Eigen vector of *F* corresponding to some Eigen value.

Proof: Let *P* be an Eigen vector of F' corresponding to the Eigen values λ,

Then $F'P = \lambda P \Rightarrow \mu F'P = \lambda \mu P$

Since $\mu F' = F\mu$, we get $F\mu P = \lambda \mu P$

Hence μP is Eigen vector of *F* corresponding to Eigen values λ.

Theorem: The necessary and sufficient condition that V_n be an almost complex manifold is that it contain a tangent bundle π_m of *dim m* and a tangent bundle $\tilde{\pi}_m$ conjugate to π_m such that $\pi_m \cap \tilde{\pi}_m = \phi$ and they span together a tangent bundle of *dim 2m*. Projections on π_m and $\tilde{\pi}_m$ being *L* and *M* given by

$$2L \overset{def}{=} I_n - iF, \quad 2M \overset{def}{=} I_n + iF$$

Proof: (Necessary) Let V_n be an almost complex manifold with almost complex structure $\{F\}$ whose Eigen values are $+i$ and $-i$. Let $\underset{x}{P}, x=1,2,....,m$ are Eigen vectors corresponding to Eigen value $+i$ and $\underset{x}{Q}, x=1,2,....,m$ are *m* linearly independent Eigen vectors conjugate to $\underset{x}{P}$ corresponding to the Eigen values $-i$. Then,

$$\overset{x}{a}\underset{x}{P} = 0 \Rightarrow \overset{x}{a} = 0 \qquad \forall x$$

and $$\overset{x}{b}\underset{x}{Q} = 0 \Rightarrow \overset{x}{b} = 0 \qquad \forall x$$

Now let $\overset{x}{c}\underset{x}{P} + \overset{x}{d}\underset{x}{Q} = 0$

Then $$\overset{x}{c}\underset{x}{\overline{P}} + \overset{x}{d}\underset{x}{\overline{Q}} = 0 \qquad \dots(1)$$

Since $F\underset{x}{P} = i\underset{x}{P}$, $\quad F\underset{x}{Q} = -i\underset{x}{Q}$

Then $$i\left\{\overset{x}{c}\underset{x}{P} - \overset{x}{d}\underset{x}{Q}\right\} = 0$$

$i \neq 0$ $$\left\{\overset{x}{c}\underset{x}{P} - \overset{x}{d}\underset{x}{Q}\right\} = 0 \qquad \dots(2)$$

From (1) and (2), we get

$$\overset{x}{c}\underset{x}{P} = 0 \text{ and } \overset{x}{d}\underset{x}{Q} = 0$$

i.e. $$\overset{x}{c} = 0 \text{ and } \overset{x}{d} = 0$$

since $\underset{x}{P} = 0$ and $\underset{x}{Q} = 0$ are linearly independent.

Thus $$\overset{x}{c}\underset{x}{P} + \overset{x}{d}\underset{x}{Q} = 0 \Rightarrow \overset{x}{c} = 0, \overset{x}{d} = 0 \qquad \forall$$

Therefore $\left\{\underset{x}{P}, \underset{x}{Q}\right\}$ is a linearly independent set.

Further, we have

$$L\underset{x}{P} = \underset{x}{P} \qquad L\underset{x}{Q} = 0$$

$$M\underset{x}{P}=0 \qquad M\underset{x}{Q}=\underset{x}{Q}$$

Because $2L\underset{x}{P}=\left(I_n-iF\right)\underset{x}{P}=\underset{x}{P}-iF\underset{x}{P}=\underset{x}{P}+\underset{x}{P}=2\underset{x}{P}$,

and similarly.

Thus we have proved that there is a tangent bundle π_m of *dim m* and there is a complex conjugate tangent bundle $\tilde{\pi}_m$ of *dim m* such that $\pi_m \cap \tilde{\pi}_m=\phi$ and they span to gather a tangent bundle of *dim 2m*. Projection on π_m and $\tilde{\pi}_m$ being *L* and *M*.

Conversely: Suppose that there is a tangent bundle π_m of *dim m* and a tangent bundle $\tilde{\pi}_m$ complex conjugate to π_m such that $\pi_m \cap \tilde{\pi}_m=\phi$ and the span together linear manifold of *dim 2m*, Let $\underset{x}{P}$ and $\underset{x}{Q}$ (complex conjugate to $\underset{x}{P}$) be *m* linearly independent vector in π_m and $\tilde{\pi}_m$ respectively. Let $\left\{\underset{x}{P},\underset{x}{Q}\right\}$ span a linear manifold of *dim 2m*, therefore $\left\{\underset{x}{P},\underset{x}{Q}\right\}$ is a linearly independent set. Let $\left\{\overset{x}{p},\overset{x}{q}\right\}$ be the inverse set of $\left\{\underset{x}{P},\underset{x}{Q}\right\}$. Then

$$I_n=\overset{x}{p}\otimes\underset{x}{P}+\overset{x}{q}\otimes\underset{x}{Q}$$

This equation yields,

$$\overset{x}{p}\left(\underset{y}{P}\right)=\delta_y^x=\overset{x}{q}\left(\underset{y}{Q}\right)$$

$$\overset{x}{p}\left(\underset{y}{Q}\right)=\overset{x}{q}\left(\underset{y}{P}\right)=0$$

Let us define,

$$F\overset{def}{=}i\left\{\overset{x}{p}\otimes\underset{x}{P}-\overset{x}{q}\otimes\underset{x}{Q}\right\}$$

$$F^2=FF=i^2\left\{\overset{x}{p}\otimes\underset{x}{P}-\overset{x}{q}\otimes\underset{x}{Q}\right\}\left\{\overset{x}{p}\otimes\underset{x}{P}-\overset{x}{q}\otimes\underset{x}{Q}\right\}$$

After solving, we get

$$F^2+I_n=0$$

Thus the manifold admits an almost complex structure.

Corollary: Prove that

(i) $L^2=L, \quad M^2=M, \quad LM=ML=0$

(ii) $FL=LF=iL, \quad FM=MF=-iM$

Corollary: Prove that, $L=\overset{x}{p}\otimes\underset{x}{P}$ and $M=\overset{x}{q}\otimes\underset{x}{Q}$.

Proof: Since $\left\{\overset{x}{p},\overset{x}{q}\right\}$ is inverse set of $\left\{\underset{x}{P},\underset{x}{Q}\right\}$, we have

$$I_n=\overset{x}{p}\otimes\underset{x}{P}+\overset{x}{q}\otimes\underset{x}{Q} \qquad \ldots(1)$$

and we also know,

$$2L=I_n-iF\,, \quad 2M=I_n+iF$$

Therefore $$L+M=I_n \qquad \ldots(2)$$

Operating (2) by F and using (1), we get

$$FL+FM=\overset{x}{p}\otimes F\underset{x}{P}+\overset{x}{q}\otimes F\underset{x}{Q}$$

$$i(L-M) = \overset{x}{p} \otimes \left(i \underset{x}{P}\right) + \overset{x}{q} \otimes \left(-i \underset{x}{Q}\right)$$

$i \neq 0$, $$L - M = \overset{x}{p} \otimes \underset{x}{P} + \overset{x}{q} \otimes \underset{x}{Q} \qquad \ldots(3)$$

From (2) and (3), we get the result.

Contravariant and covariant almost analytic vectors

Definition: A vector field V is said to e contravariant almost analytic if it satisfies

$$L_V F = 0$$

i.e. Lie derivatives of F with respect to V vanishes. A vector field V is said to be strictly contravariant almost analytic, if both V and $\overline{V}$ are contravariant almost analytic i.e.

$$L_V F = 0 \text{ And } L_{\overline{V}} F = 0.$$

Lemma: We have on an almost complex manifold,

(i) $L_{\overline{V}} F = L_V F + N(V, X)$

Equivalent to,

(ii) $L_{\overline{V}} F + L_V F = \overline{N(V, X)}$

(iii) $(L_{\overline{V}} F)(X) = \overline{(L_V F)(\overline{X})} + N(V, \overline{X})$

(iv) $\overline{(L_{\overline{V}} F)(\overline{X})} + (L_V F)(\overline{X}) = \overline{N(V, \overline{X})}$

Proof: Consider,

$$L_{\overline{V}} \overline{X} = (L_{\overline{V}} F)(X) + F(L_{\overline{V}} X)$$

or $$[\overline{V}, \overline{X}] = (L_{\overline{V}} F)(X) + \overline{[\overline{V}, X]} \qquad \ldots(1)$$

Further taking Lie derivative of $\overline{X}$ with respect to V, we get

$$L_V \overline{X} = (L_V F)(X) + F(L_V X)$$

or $$[V, \overline{X}] = \overline{(L_V F)(X)} - [V, X] \qquad \ldots(2)$$

From (1) and (2), we have

$$(L_{\overline{V}} F)(X) - \overline{(L_V F)(X)} = N(V, X) \qquad \ldots(3)$$

Where $$N(V, X) \overset{def}{=} [\overline{V}, \overline{X}] - [V, X] - \overline{[\overline{V}, X]} - \overline{[V, \overline{X}]}$$

Barring equation (3), we get

$$\overline{(L_{\overline{V}} F)(X)} + (L_V F)(X) = \overline{N(V, X)} \qquad \ldots(4)$$

From (3) and (4) we get results.

Theorem: A necessary and sufficient condition that vector field V on and almost complex manifold be contravariant almost analytic is

$$L_V \overline{X} = \overline{L_V X} \Rightarrow [V, \overline{X}] = \overline{[V, X]}$$

Proof: A vector field V is contravariant almost analytic if

$$L_V F = 0 \qquad \ldots(1)$$

$$L_V \overline{X} = (L_V F)(X) + \overline{L_V X}$$

Using (1) in above equation, we get

$$L_V \overline{X} = \overline{L_V X}.$$

Theorem: Lie derivative of Nijenhuis tensor with respect to a contravariant almost analytic vector V, on an almost complex manifold vanishes, i.e.

$$L_V N = 0.$$

Definition: A *1-from* ω is said to be covariant almost analytic if it satisfies,

$$\omega(((D_X F)Y) - (D_Y F)X) = (D_{\bar{X}}\omega)(Y) - (D_X\omega)(\bar{Y})$$

Where D is a symmetric connection in V_n.

Theorem: If a *1-from* ω is covariant almost analytic then $d\omega$ is pure in both the slots, i.e.

$$(d\omega)(\bar{X},\bar{Y}) + (d\omega)(X,Y) = 0.$$

Proof: Since,

$$(d\omega)(X,Y) = (D_X\omega)(Y) - (D_Y\omega)(X) \quad \ldots(1)$$

Using definition,

$$\omega((D_X F)(Y) - (D_Y F)(X)) = (D_{\bar{X}}\omega)(Y) - (D_X\omega)(\bar{Y}) \quad \ldots(2)$$

and

$$\omega((D_Y F)(X) - (D_X F)(Y)) = (D_{\bar{Y}}\omega)(X) - (D_Y\omega)(\bar{X}) \quad \ldots(3)$$

Adding (2) and (3) then barring Y, we get the result.

Cor.: If $\tilde{\omega}(X) \overset{def}{=} \omega(\bar{X}) \Leftrightarrow \tilde{\omega}(\bar{X}) = -\omega(X)$

Then $d\tilde{\omega}(X,Y) = d\omega(\bar{X},Y)$

Equivalent to $d\tilde{\omega}(\bar{X},Y) + d\omega(X,Y) = 0$.

Proof: We have from definition,

$$\tilde{\omega}(Y) = \omega(\bar{Y})$$

Taking covariant derivative with respect to *X*, we get

$$(D_X\tilde{\omega})(Y) = (D_X\omega)(\bar{Y}) + \omega((D_X F)(Y)) \quad \ldots(1)$$

since ω is covariant almost analytic, we have

$$\omega((D_X F)(Y) - (D_Y F)(X)) = (D_{\bar{X}}\omega)(Y) - (D_X\omega)(\bar{Y}) \quad \ldots(2)$$

From (1), we have

$$(D_Y\tilde{\omega})(X) = (D_Y\omega)(\bar{X}) + \omega((D_Y F)(X)) \quad \ldots(3)$$

From (1) and (3), we have

$$(D_X\tilde{\omega})(Y) - (D_Y\tilde{\omega})(X) = (D_X\omega)(\bar{Y}) - (D_Y\omega)(\bar{X}) + \omega((D_X F)(Y)) - \omega((D_Y F)(X)) \quad \ldots(4)$$

Using (2) in (4), we get

$$(D_X\tilde{\omega})(Y) - (D_Y\tilde{\omega})(X) = (D_{\bar{X}}\omega)(Y) - (D_Y\omega)(\bar{X}) \quad \ldots(5)$$

Since,

$$(d\omega)(X,Y) = (D_X\omega)(Y) - (D_Y\omega)(X) \quad \ldots(6)$$

From (5) and (6) we get the result.

Theroem: If *1-from* ω is covariant almost analytic on an almost complex manifold then $\tilde{\omega}$ is also covariant analytic.

Where $\tilde{\omega}(X) \overset{def}{=} \omega(\bar{X}) \Leftrightarrow \tilde{\omega}(\bar{X}) = -\omega(X)$

Proof: Since *1-from* ω is covariant almost analytic, then we have

$$\omega((D_X F)(Y) - (D_Y F)(X)) \overset{def}{=} (D_{\bar{X}}\omega)(Y) - (D_X\omega)(\bar{Y}) \quad \ldots(1)$$

Taking covariant derivative of $\tilde{\omega}(Y) = \omega(\bar{Y})$ with respect to *X* and $\bar{X}$, we get

$$(D_X\tilde{\omega})(Y) = (D_X\omega)(\bar{Y}) + \omega((D_X F)(Y)) \quad \ldots(2)$$

and

$$(D_{\bar{X}}\tilde{\omega})(Y) = (D_{\bar{X}}\omega)(\bar{Y}) + \omega((D_{\bar{X}} F)(Y)) \quad \ldots(3)$$

Barring *Y* in (2)

$$(D_X\tilde{\omega})(\bar{Y}) = -(D_X\omega)(Y) + \omega((D_X F)(\bar{Y})) \quad \ldots(4)$$

Now consider

$$F(\overline{Y}) = -Y$$

Taking its covariant derivative with respect to *X*, we get

$$(D_X F)(\overline{Y}) = -F((D_X F)(Y)) \quad \ldots(5)$$

Operating by ω, we get

$$\omega((D_X F)(\overline{Y})) = -\tilde{\omega}((D_X F)(X)) \quad \ldots(6)$$

Putting (6) in (4), we get

$$(D_X \tilde{\omega})(\overline{Y}) = -(D_X \omega)(Y) - \tilde{\omega}((D_X F)(Y)) \quad \ldots(7)$$

Now using (3) and (7), we get

$$(D_X \tilde{\omega})(Y) - (D_X \tilde{\omega})(\overline{Y}) = (D_{\overline{X}} \omega)(\overline{Y}) + (D_X \omega)(Y) + \omega((D_{\overline{X}} F)(Y)) + \omega((D_{\overline{X}} F)(Y)) + \tilde{\omega}((D_X F)(Y)) \quad \ldots(8)$$

Interchanging *X* and *Y* in (1), then barring *X*, we get

$$-\tilde{\omega}((D_Y F)(X)) = (D_{\overline{X}} \omega)(\overline{Y})) + (D_X \omega)(Y) + \omega((D_{\overline{X}} F)(Y)) \quad \ldots(9)$$

Using (9) in (8), we get the result.

Theorem: If on an almost complex manifold, the covariant almost analytic vector field ω is closed then $\tilde{\omega}$ is also closed.

Where $\tilde{\omega}(X) \overset{def}{=} \omega(\overline{X}) \Leftrightarrow \tilde{\omega}(\overline{X}) = -\omega(X)$

Proof: We have

$$d\tilde{\omega}(X, Y) = d\omega(\overline{X}, Y)$$

or

$$d\tilde{\omega}(\overline{X}, Y) = -d\omega(X, Y)$$

If ω is closed, then $d\omega = 0 \Rightarrow (d\tilde{\omega})(\overline{X}, Y) = 0 \Rightarrow d\tilde{\omega} = 0$.

Theorem: If ω and $\tilde{\omega}$ are both closed on an almost complex manifold then they are both covariant almost analytic, where $\tilde{\omega}(X) \overset{def}{=} \omega(\overline{X}) \Leftrightarrow \tilde{\omega}(\overline{X}) = -\omega(X)$.

Proof: If ω and $\tilde{\omega}$ are both closed then

$$d\omega(X, Y) = (D_X \omega)(Y) - (D_Y \omega)(X) = 0 \quad \ldots(1)$$

and

$$d\tilde{\omega}(X, Y) = (D_X \tilde{\omega})(Y) - (D_Y \tilde{\omega})(X) = 0 \quad \ldots(2)$$

Now consider,

$$\tilde{\omega}(Y) = \omega(\overline{Y})$$

Taking its covariant derivative with respect to *X*, we get

$$(D_X \tilde{\omega})(Y) = (D_X \omega)(\overline{Y}) + \omega((D_X F)(Y)) \quad \ldots(3)$$

Interchanging *X* and *Y*, we get

$$(D_Y \tilde{\omega})(X) = (D_Y \omega)(\overline{X}) + \omega((D_Y F)(X)) \quad \ldots(4)$$

From (3) and (4), we have

$$(D_X \tilde{\omega})(Y) - (D_Y \tilde{\omega})(X) = (D_X \omega)(\overline{Y}) - (D_Y \omega)(\overline{X}) + \omega((D_X F)(Y)) - \omega((D_Y F)(X))$$

or

$$(D_Y \omega)(\overline{X}) - (D_X \omega)(\overline{Y}) = \omega((D_X F)(Y) - (D_Y F)(X))$$

Now

$$(D_{\overline{X}} \omega)(Y) - (D_X \omega)(Y) - (D_{\overline{X}} \omega)(Y) + (D_Y \omega)(\overline{X}) = \omega((D_X F)(Y) - (D_Y F)(X))$$

Using (1), we get

$$(D_{\overline{X}} \omega)(Y) - (D_X \omega)(\overline{Y}) = \omega((D_X F)(Y) - (D_Y F)(X))$$

Hence *1-from* ω is covariant almost analytic.

We know that if *1-from* ω is covariant almost analytic then $\tilde{\omega}$ is also covariant almost analytic.

F-Connection

Def.: An affine connection *D* on an almost complex manifold is called an *F-connection* if,

$$(D_X F)(Y) = 0 \Leftrightarrow D_X \overline{Y} = \overline{D_X Y}$$

In an almost complex manifold, we have

$$\overline{D_X \overline{Y}} + D_X Y = 0.$$

Theorem: Given an arbitrary connection *B*, connection *D* is defined by,

$$2D_X Y \overset{def}{=} B_X Y - \overline{B_X \overline{Y}}.$$

Then show that *D* is an *F-connection.*

Proof: We have,

$$2D_X Y = B_X Y - \overline{B_X \overline{Y}} \qquad \ldots(1)$$

Barring Y in (1), we get

$$2D_X \overline{Y} = B_X \overline{Y} + \overline{B_X Y} \qquad \ldots(2)$$

Barring whole equation (1), we get

$$2\overline{D_X Y} = \overline{B_X Y} + B_X \overline{Y} \qquad \ldots(3)$$

From (2) and (3), we get the result.

Theorem: On an almost complex manifold if the *F-connection D* is symmetric then Nijenhuis tensor vanishes.

Proof: Nijenhuis tensor on an almost complex manifold is defined as

$$N(X,Y) = [F,F](X,Y) \overset{def}{=} [\overline{X},\overline{Y}] - [X,Y] - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]}.$$

When connection D is symmetric F-connection,

(i) Torson tensor $s = 0$,

(ii) $D_X F = 0$

Where

$$s(X,Y) \overset{def}{=} D_X Y - D_Y X - [X,Y]$$

Since $s = 0$

Therefore $D_X Y - D_Y X = [X,Y]$

$$N(X,Y) = D_{\overline{X}} \overline{Y} - D_{\overline{Y}} \overline{X} - D_X Y + D_Y X - \overline{D_{\overline{X}} Y} + \overline{D_Y \overline{X}} - \overline{D_X \overline{Y}} + \overline{D_{\overline{Y}} X} \qquad \ldots(1)$$

Since D is an F-connection,

$$D_X F = 0 \Rightarrow D_X \overline{Y} = \overline{D_X Y}. \qquad \ldots(2)$$

Using (2) in (1), we get the result.

References

[1] A.Z. Petrov (1969): Einstein Spaces Pergamon Press, Oxford.

[2] Hicks, N.J. (1965): Notes on Differential Geometry D. Van Nostrand Company Inc., New York.

[3] K. Yano (1965): Differential geometry on Complex and Almost Complex Spaces Pergmon Press.

[4] K. Yano and M. Kon (1984): Structures on Manifolds World Scientific Publishing Company Pvt. Ltd., P.O. Box 128, Farrer Road, Singapore, 9128.

[5] R.S. Mishra (1984): Structures on a differentiable manifold and their applications Chandrama Prakashan, 50A Balrampur House, Allahabad, India.

[6] Lovejoy S. Das and Ram Nivas: Harmonic morphism an almost r-contact metric manifolds Algebras Groups and Geometries, 22 (2005), 61-68.

[7] Lovejoy S. Das and Ram Nivas: On certain structures defined on the tangent bundle Rocky Mountain J. of Mathematics, U.S.A., vol. 36, no. 6 (2006), pp. 1857-1866.