Inertia of Distance Matrix of Neuron Graph Distance Matrix

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Abstract :-

Let D denote the distance matrix of a connected graph G. The inertia of D is the triple of integers $(n_+(D), n_-(D), n_0(D))$, where $n_+(D), n_-(D), n_0(D)$ denote the number of positive, negative and 0 eigenvalues of D, respectively. In this paper, we will find the inertia of distance matrix of spider graph which is a extension of wheel graph. [1]

1. Introduction :-

Let G be a undirected connected graph with n vertices. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, then the distance between two vertices v_i and v_j is the length of shortest path between v_i and v_j, denoted by $d_G(v_i, v_j)$. The distance matrix of a graph is defined in a similar way as the adjacency matrix: the entry in the i^{th} row, j^{th} th column is the distance between the i^{th} and j^{th} vertex. In this paper we will denote distance matrix of garph G by G only. The $D-eigenvalues$ of a graph G are the eigenvalues of its distance matrix G which form the distance spectrum or $D-spectrum$ of G.

The inertia of a real symmetric matrix G is triple (x, y, z) where x, y, z are the number of positive, negative and zero eigenvalues of distance matrix of a graph G, respectively. It is denoted by $In(G) = (x, y, z)$.

Definition :- Inertia of distance matrix of an undirected connected graph G

Let G be an undirected connected graph. Then inertia of G is triple (x, y, z) where x, y, z are the number of positive, negative and zero eigenvalues of distance matrix of a graph G, respectively. It is denoted by $In(G) = (x, y, z)$.

Definition :- Neuron of tree T, $\aleph(T)$

Let T be a tree on $r+1$ vertices. Then $\aleph(T)$ is a Neuron form by replacing edges of T by connected graph G_i, for $1 \leq i \leq r$.

We say $\aleph(T) = G$ is Neuron with r blocks, labled by $G_1, G_2, \ldots, G_r$.

Definition :- Lotus of star S, $\Psi(S)$

Let S be a star on $r+1$ vertices. Then $\Psi(S)$ is a lotus form by replacing edges of S by connected graph G_i, for $1 \leq i \leq r$.

We say $\Psi(S) = G$ is lotus with r blocks, labled by $G_1, G_2, \ldots, G_r$.

Every Lotus is a Neuron as every star is a tree.

Sylvester Theorem :-

Let A and B be two symmetric matrices. Then there exists an invertible matrix S such that $A = S \cdot B \cdot S'$ if and only if $In(A) = In(B)$, where S' is a transpose of S.

Cauchy's Interlacing Theorem :-

Let A be a Hermitian matrix of order n and B be a principal submatrix of A of order m. If $\lambda_n \leq \lambda_{n-1} \leq \lambda_{n-2} \leq \cdots \leq \lambda_2 \leq \lambda_1$ lists the eigenvalues of A and $\mu_m \leq \mu_{m-1} \leq \mu_{m-2} \leq \cdots \leq \mu_2 \leq \mu_1$ lists the eigenvalues of B. Then $\lambda_{i+n-m} \leq \mu_i \leq \lambda_i$, where $i = 1, 2, \ldots, m$.
In perticular, if $m = n-1$, then, $\lambda_{i+1} \leq \mu_i \leq \lambda_i$, where $i = 1, 2, \ldots, n-1$.

Notations :-

(1) We will denote distance matrix of graph G by G only.
(2) If G and H be any two connected graph, then $Gu \cdot vH$ is graph obtain by merging vertex u of G with vertex v of H.
(3) Matrix $\widetilde{G}_u$ is a matrix formed by removing row and column of G corresponding to vertex u, after subtracting it from remaining rows and columns respectively.
(4) $n_+(G), n_-(G)$ and $n_0(G)$ are the number of positive, negative and zero eigenvalue of G respectively.
(5) $d_G(a, b) =$ length of shortest path from vertex a to vertex b in graph G.
(6) We will denote transpose of matrix A by A'.

Theorem 1:-

Let G & H be any two connected graphs. Let $Gu \cdot vH$ be a graph formed by merging vertex u of graph G and vertex v of graph H. Then $In((\widetilde{Gu \cdot vH})_u) = In(\widetilde{G}_u) + In(\widetilde{H}_u)$

Proof :-

Let $V(G) = \{u = u_1, u_2, \ldots, u_{n+1}\}$ & $V(H) = \{v = v_1, v_2, \ldots, v_{m+1}\}$
$\therefore V(Gu \cdot vH) = \{u_{n+1}, u_n, \ldots, u_1 = u = v = v_1, v_2, \ldots, v_{m+1}\}$
$\therefore |V(Gu \cdot vH)| = n + m + 1$
Since $u = v$ is the only common vertex of G and H.
$\therefore d_{Gu \cdot vH}(u_i, v_j) = d_G(u_i, u) + d_H(v_j, u)$.

Consider distance matrix of $Gu \cdot vH$.
Where, $d(u, G-u) = \begin{bmatrix} d_G(u, u_2) & \cdots & d_G(u, u_{n+1}) \end{bmatrix}$, $d(u, H-u) = \begin{bmatrix} d_H(v, v_2) & \cdots & d_H(v, v_{m+1}) \end{bmatrix}$.
G_u and H_u is principal submatrix of distance of matrix G and H respectively, corresponding to u.
$L_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{1 \times n}$.

Let $Q = \begin{bmatrix} I_n & -L_n' & 0 \\ 0 & 1 & 0 \\ 0 & -L_m' & I_m \end{bmatrix}_{(n+1+m)\times(n+1+m)}$

Where I_n is a identity matrix of order n.

Consider $\widetilde{Gu \cdot vH} = Q \cdot (Gu \cdot vH) \cdot Q' =$

$$\begin{bmatrix} I_n & -L_n' & 0 \\ 0 & 1 & 0 \\ 0 & -L_m' & I_m \end{bmatrix} \times \begin{bmatrix} G_u & d(u,G-u)' & d(u,G-u)' \cdot L_m + L_n' \cdot d(u,H-u) \\ d(u,G-u) & 0 & d(u,H-u) \\ L_m' \cdot d(u,G-u) + d(u,H-u)' \cdot L_n & d(u,H-u)' & H_u \end{bmatrix} \times Q'$$

$$= \begin{bmatrix} G_u - L_n' \cdot d(u,G-u) & d(u,G-u)' & d(u,G-u)' \cdot L_m \\ d(u,G-u) & 0 & d(u,H-u) \\ d(u,H-u)' \cdot L_n & d(u,H-u)' & H_u - L_m' \cdot d(u,H-u) \end{bmatrix} \times \begin{bmatrix} I_n & 0 & 0 \\ -L_n & 1 & -L_m \\ 0 & 0 & I_m \end{bmatrix}$$

$$= \begin{bmatrix} G_u - d(u,G-u)' \cdot L_n - L_n' \cdot d(u,G-u) & d(u,G-u)' & 0 \\ d(u,G-u) & 0 & d(u,H-u) \\ 0 & d(u,H-u)' & H_u - d(u,H-u)' \cdot L_m - L_m' \cdot d(u,H-u) \end{bmatrix}$$

Note that $\widetilde{G_u} = G_u - d(u,G-u)' \cdot L_n - L_n' \cdot d(u,G-u)$ and $\widetilde{H_u} = H_u - d(u,H-u)' \cdot L_m - L_m' \cdot d(u,H-u)$

$\therefore \widetilde{Gu \cdot vH} =$

$$\therefore (\widetilde{Gu \cdot vH})_u = \begin{bmatrix} \widetilde{G_u} & 0 \\ 0 & \widetilde{H_u} \end{bmatrix}$$

$\therefore In((\widetilde{Gu \cdot vH})_u) = In(\widetilde{G}_u) + In(\widetilde{H}_u)$ ...□□□

Corollary 1.1:-

Let G be Neuron with blocks $G_1, G_2, \cdots, G_r$. Let $U = \{u : u \in V(G_i) \cap V(G_j), 1 \le i < j \le r\}$.
Then, $In(\widetilde{G}_u) = \sum_{i=1}^{r} In(\widetilde{(G_i)}_u)$, for $u \in U$.

Proof :-

We will prove this result by induction on $r \ge 2$ (number of blocks).
Let $r = 2$. That is $G = G_1 \cdot G_2$
Let u be a merging vertex of G_1 and G_2.
By Theorem 1, we get result, $In((\widetilde{G_1 \cdot G_2})_u) = In(\widetilde{G_{1u}}) + In(\widetilde{G_{2u}})$
Hence result is true for $r = 2$
Assume result is true for number of blocks less than r.
Let $u \in U$.
$\therefore \exists\, p, q$ for $1 \le p < q \le r$ such that $u \in V(G_p) \cap V(G_q)$.
Since G is cactus, therfore $\exists$ subgraph H_1 and H_2 such that $G = H_1 \cdot H_2$ merged at u and G_p, G_q are subgraph of H_1 and H_2 respectively.

W.L.O.G

Let H_1 **is a cactus containing blocks** $G_1, G_2, \ldots, G_k$ **and** H_2 **is a cactus containing blocks** $G_{k+1}, G_{k+2}, \ldots, G_r$, **where** $1 \leq p \leq k < q \leq r$

$\therefore$ **by induction hypotheses,** $In(\widetilde{(H_1)}_u) = \sum_{i=1}^{k} In(\widetilde{(G_i)}_u)$ **and** $In(\widetilde{(H_2)}_u) = \sum_{i=k+1}^{r} In(\widetilde{(G_i)}_u)$

Also $G = H_1 \cdot H_2$,

$\therefore$ **by theorem 1 we get,**

$$In(\widetilde{G}_u) = In(\widetilde{(H_1 \cdot H_2)}_u)$$
$$= In(\widetilde{(H_1)}_u) + In(\widetilde{(H_2)}_u)$$
$$= \sum_{i=1}^{k} In(\widetilde{(G_i)}_u) + \sum_{i=k+1}^{r} In(\widetilde{(G_i)}_u)$$
$$= \sum_{i=1}^{r} In(\widetilde{(G_i)}_u)$$

$\therefore$ **by induction, result is true for** r **blocks.**

Hence proved. ...□□□

Theorem 2:-

Let G **&** H **be any two connected graphs such that** $In(G) = (1, y_1, z_1)$, $In(\widetilde{G}_u) = (0, y_1, z_1)$, $In(H) = (1, y_2, z_2)$ **and** $In(\widetilde{H}_u) = (0, y_2, z_2)$**. Let** $Gu \cdot vH$ **be a graph formed by merging vertex** u **of graph** G **and vertex** v **of graph** H**. Then** $In(Gu \cdot vH) = (1, y_1 + y_2, z_1 + z_2)$

Proof :-

Let $V(G) = \{u = u_1, u_2, \ldots, u_{n+1}\}$ **&** $V(H) = \{v = v_1, v_2, \ldots, v_{m+1}\}$.

$\therefore V(Gu \cdot vH) = \{u_{n+1}, u_n, \ldots, u_1 = u = v = v_1, v_2, \ldots, v_{m+1}\}$.

$\therefore |V(Gu \cdot vH)| = n + m + 1$.

(1) To prove, $n_+(Gu \cdot vH) = 1$.

We have $In(\widetilde{(Gu \cdot vH)}_u) = In(\widetilde{G}_u) + In(\widetilde{H}_u)$ ...**by theorem 1**

$\therefore$ **By given condition,** $In(\widetilde{(Gu \cdot vH)}_u) = (0, y_1, z_1) + (0, y_2, z_2) = (0, y_1 + y_2, z_1 + z_2)$

$\therefore n_+(\widetilde{(Gu \cdot vH)}_u) = 0$, $n_0(\widetilde{(Gu \cdot vH)}_u) = z_1 + z_2$ **and** $n_-(\widetilde{(Gu \cdot vH)}_u) = y_1 + y_2$ ...**(2.1)**

$\therefore n_+(Gu \cdot vH) \leq 1$ ...**by interlacing theorem**

But $Trace(Gu \cdot vH) = 0 \implies n_+(Gu \cdot vH) \geq 1$

$\therefore n_+(Gu \cdot vH) = 1$ ...**(*)**

(2) To prove, $n_0(Gu \cdot vH) = z_1 + z_2$.

Since $n_+(Gu \cdot vH) = 1$.

$\therefore$ **By interlacing theorem** $n_0(Gu \cdot vH) \leq n_0(\widetilde{(Gu \cdot vH)}_u)$

$\therefore$ **By (2.1)** $n_0(Gu \cdot vH) \leq z_1 + z_2$ ...**(2.2)**

Claim:- $z_1 + z_2 \leq n_0(Gu \cdot vH)$.

Case (i):- $z_2 = 0$.

Since G **be induce subgraph of** $Gu \cdot vH$.

$\therefore$ **by interlacing theorem,** $n_0(G) \leq n_0(Gu \cdot vH)$.

$\therefore z_1 \leq n_0(Gu \cdot vH)$ ...**given** $n_0(G) = z_1$

$\therefore z_1 + z_2 \leq n_0(Gu \cdot vH)$ ...**(2.3)**

Case (ii):- $z_2 \neq 0$.

Consider distance matrix of $Gu \cdot vH$.

$$= \begin{bmatrix} G_u & d(u, G-u)' & d(u, G-u)' \cdot L_m + L_n' \cdot d(u, H-u) \\ d(u, G-u) & 0 & d(u, H-u) \\ L_m' \cdot d(u, G-u) + d(u, H-u)' \cdot L_n & d(u, H-u)' & H_u \end{bmatrix}_{(n+1+m) \times (n+1+m)}$$

Where, $d(u, G-u) = \begin{bmatrix} d_G(u, u_2) & \cdots & d_G(u, u_{n+1}) \end{bmatrix}$, $d(u, H-u) = \begin{bmatrix} d_H(v, v_2) & \cdots & d_H(v, v_{m+1}) \end{bmatrix}$.

G_u and H_u is principal submatrix of distance of matrix G and H respectively, corresponding to u and $L_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{1\times n}$

Let $Q_n = \begin{bmatrix} 1 & 0 \\ -L'_{n-1} & I_{n-1} \end{bmatrix}_{n\times n}$

Where I_n is a identity matrix of order n. Note that, $det(Q_n) = 1$.

Hence by Sylvester theorem, for any symmetric matrix A, $In(Q_n \cdot A \cdot Q'_n) = In(A)$ **. . . (2.4)**

Let $R_{n,m} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -L'_m & I_m \end{bmatrix}_{(n+1+m)\times(n+1+m)}$ **. Note that,** $det(R_{n,m}) = 1$.

Consider, $(R_{n,m})(Gu \cdot vH)(R_{n,m})' =$

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -L'_m & I_m \end{bmatrix} \begin{bmatrix} G_u & d(u,G-u)' & d(u,G-u)' \cdot L_m + L'_n \cdot d(u,H-u) \\ d(u,G-u) & 0 & d(u,H-u) \\ L'_m \cdot d(u,G-u) + d(u,H-u)' \cdot L_n & d(u,H-u)' & H_u \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & 1 & -L_m \\ 0 & 0 & I_m \end{bmatrix}$$

$$= \begin{bmatrix} G_u & d(u,G-u)' & d(u,G-u)' \cdot L_m + L'_n \cdot d(u,H-u) \\ d(u,G-u) & 0 & d(u,H-u) \\ d(u,H-u)' \cdot L_n & d(u,H-u)' & H_u - L'_m \cdot d(u,H-u) \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & 1 & -L_m \\ 0 & 0 & I_m \end{bmatrix}$$

$$= \begin{bmatrix} G_u & d(u,G-u)' & L'_n \cdot d(u,H-u) \\ d(u,G-u) & 0 & d(u,H-u) \\ d(u,H-u)' \cdot L_n & d(u,H-u)' & \widetilde{H_u} \end{bmatrix}_{(n+1+m)\times(n+1+m)}$$

Where,

We have $Q_{m+1} \cdot H \cdot Q'_{m+1} = \begin{bmatrix} 0 & d(u,H-u) \\ d(u,H-u)' & \widetilde{H_u} \end{bmatrix}_{(m+1)\times(m+1)} = H^*_u$ **(say)**

$\therefore$ **by (2.4)** $In(H) = In(Q_{m+1} \cdot H \cdot Q'_{m+1}) = In(H^*_u)$

$\therefore Nullity(H^*_u) = Nullity(H) = z_2 \geq 1$ **. . . H^*_u, H are symmetric**

W.L.O.G. $\exists$ z_2 row transformations which vanishes rows corresponding to $v_{m+2-z_2}, v_{m+3-z_2}, \ldots, v_{m+1}$ in H^*_u.

Since $Nullity(H^*_u) = z_2 = Nullity(\widetilde{H_u})$ and $\widetilde{H_u}$ is principle submatrix of H^*_u corresponding to first row.

$\therefore$ The above row transformations are independent of first row of H^*_u, i.e. R_{n+1}

Consider a submtrix A of order $m \times (n+1+m)$ which contain last m rows of $(R_{n,m})(Gu \cdot vH)(R_{n,m})'$.

From (2.5), we get, first n column of A are same as the $(n+1)^{th}$ column of A , which is a first column of H^*_u excluding first entry(0).

$\therefore$ Above row transformation also vanishes last z_2 entries of first n column of A.

Let the row transformation corresponding to v_{m+1} is $R_{n+m+1} = b_{(n+m+1)(n+2)}R_{n+2} + b_{(n+m+1)(n+3)}R_{n+3} + \cdots + b_{(n+m+1)(n+m)}R_{n+m} + b_{(n+m+1)(n+m+1)}R_{n+m+1}$, **where** $b_{(n+m+1)(n+m+1)} \neq 0$

Let P_{n+m+1} be the corresponding row transformation matrix.

$\therefore P_{n+m+1} =$

Here, $det(P_{n+m+1}) = b_{(n+m+1)(n+m+1)} \neq 0$. **. . .** $\because d(u,H-u) > 0$

By using (2.5), we get $P_{n+m+1} \cdot (R_{n,m})(Gu \cdot vH)(R_{n,m})' \cdot P'_{n+m+1} =$

= **Similarly, let the row transformation corresponding to v_i is** $R_{n+i} = b_{(n+i)(n+2)}R_{n+2} + b_{(n+i)(n+3)}R_{n+3} + \cdots + b_{(n+i)(n+i-1)}R_{n+i-1} + b_{(n+i)(n+i)}R_{n+i}$, **where** $b_{(n+i)(n+i)} \neq 0$, **for** $i = m+2-z_2, m+3-z_2, \ldots, m+1$.

And P_{n+i} be the corresponding row transformation matrix.

$\therefore P_{n+m+2-z_2} \cdot P_{n+m+3-z_2} \cdots P_{n+m} \cdot P_{n+m+1} \cdot (R_{n,m})(Gu \cdot vH)(R_{n,m})' \cdot P'_{n+m+1} \cdot P'_{n+m} \cdots P'_{n+m+3-z_2} \cdot P'_{n+m+2-z_2} =$

Let S be the principle submatrix of $Gu \cdot vH$, formed by removing last z_2 rows and columns. Note that S is also the principle submatrix of $(R_{n,m})(Gu \cdot vH)(R_{n,m})'$, formed by removing last z_2 rows and columns.

$\therefore P_{n+m+2-z_2} \cdot P_{n+m+3-z_2} \ldots P_{n+m} \cdot P_{n+m+1} \cdot (R_{n,m})(Gu \cdot vH)(R_{n,m})' \cdot P'_{n+m+1} \cdot P'_{n+m} \ldots P'_{n+m+3-z_2} \cdot P'_{n+m+2-z_2} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$

Let $R = P_{n+m+2-z_2} \cdot P_{n+m+3-z_2} \ldots P_{n+m} \cdot P_{n+m+1} \cdot (R_{n,m})$

$\therefore R \cdot (Gu \cdot vH) \cdot R' = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$
$\therefore In(R \cdot (Gu \cdot vH) \cdot R') = In(S) + (0, 0, z_2)$
Since $det(P_i) \neq 0$ **and** $det(R_{n,m}) = 1 \implies det(R) \neq 0 \implies R$ **is invertible.**
$\therefore$ **By Sylvester theorem,** $In(Gu \cdot vH) = In(R \cdot (Gu \cdot vH) \cdot R') = In(S) + (0, 0, z_2)$
$\therefore n_0(Gu \cdot vH) = n_0(S) + z_2$ **. . . (2.6)**

Note that G **is principle submatrix of** S.
$\therefore n_0(G) \leq n_0(S)$ **. . . by Interlacing theorem**
i.e. $z_1 \leq n_0(S)$
$\therefore z_1 + z_2 \leq n_0(S) + z_2 \implies z_1 + z_2 \leq n_0(Gu \cdot vH)$. **. . . By (2.6)**
$\therefore$ **by both the cases, we get,** $z_1 + z_2 \leq n_0(Gu \cdot vH)$. **. . . (2.7)**

$\therefore$ **By (2.2) and (2.7), we get,** $n_0(Gu \cdot vH) = z_1 + z_2$ **. . . (**)**

(3) **To prove** $n_-(Gu \cdot vH) = y_1 + y_2$.

We have $|V(Gu \cdot vH)| = |V(G)| + |V(H)| - 1$
$\therefore n_+(Gu \cdot vH) + n_-(Gu \cdot vH) + n_0(Gu \cdot vH) = (1 + y_1 + z_1) + (1 + y_2 + z_2) - 1$
$\therefore (1) + n_-(Gu \cdot vH) + (z_1 + z_2) = 1 + (y_1 + y_2) + (z_1 + z_2)$ **. . . by (*) and (**)**
$\therefore n_-(Gu \cdot vH) = y_1 + y_2$
Hence proved. **. . .□□□**

Theorem 3:-

Let G be Neuron with blocks $G_1, G_2, \cdots, G_r$. **Let** $U = \{u : u \in V(G_i) \cap V(G_j), 1 \leq i < j \leq r\}$. **If** $In(G_i) = (1, y_i, z_i)$ **and** $In((\widetilde{G_i})_u) = (0, y_i, z_i)$, **where,** $u \in U$
Then, $In(G) = (1, \sum_{i=1}^{r} n_i, \sum_{i=1}^{r} z_i)$.

Proof :-

We will prove this result by induction on $r \geq 2$ **(number of blocks).**
Let $r = 2$. **That is** $G = G_1 \cdot G_2$
Let $In(G_1) = (1, y_1, z_1)$ **and** $In(G_2) = (1, y_2, z_2)$
Let u **be a merging vertex of** G_1 **and** G_2.
By Theorem 2, we get result, $In(G_1 \cdot G_2) = (1, y_1 + y_2, z_1 + z_2)$
Hence result is true for $r = 2$

Assume result is true for number of blocks less than r.
Let $u \in U$.
$\therefore \exists\, p, q$ **for** $1 \leq p < q \leq r$ **such that** $u \in V(G_p) \cap V(G_q)$.
Since G **is Neuron, therfore** $\exists$ **subgraph** H_1 **and** H_2 **of** G **such that** $G = H_1 \cdot H_2$ **merged at** u **and** G_p, G_q **are subgraph of** H_1 **and** H_2 **respectively.**
W.L.O.G
Let H_1 **is a Neuron containing blocks** $G_1, G_2, \ldots, G_k$ **and** H_2 **is a Neuron containing blocks** $G_{k+1}, G_{k+2}, \ldots, G_r$, **where** $1 \leq p \leq k < q \leq r$

$\therefore$ **by induction hypotheses,** $In(H_1) = (1, \sum_{i=1}^{k} y_i, \sum_{i=1}^{k} z_i)$ **and** $In(H_2) = (1, \sum_{i=k+1}^{r} y_i, \sum_{i=k+1}^{r} z_i)$

Also by Corollary 1.1, $In((\widetilde{H_1})_u) = (0, \sum_{i=1}^{k} y_i, \sum_{i=1}^{k} z_i)$ **and** $In((\widetilde{H_2})_u) = (0, \sum_{i=k+1}^{r} y_i, \sum_{i=k+1}^{r} z_i)$

Note that, $G = H_1 \cdot H_2$, **merged at** u.
$\therefore$ **by theorem 2 we get,**
$In(G) = In(H_1 \cdot H_2)$
$= (1, \sum_{i=1}^{k} y_i + \sum_{i=k+1}^{r} y_i, \sum_{i=1}^{k} z_i + \sum_{i=k+1}^{r} z_i)$
$= (1, \sum_{i=1}^{r} y_i, \sum_{i=1}^{r} z_i)$

$\therefore$ by induction, result is true for r blocks.
Hence proved. ...□□□

Corollary 3.1:-

Let T be tree on n vertices. Then $In(T) = (1, n-1, 0)$.

Proof :-

Note that Tree is cactus having $n-1$ blocks K_2.
We have $In(K_2) = (1,1,0)$ and $In(\widetilde{K_2}) = (0,1,0)$.

$\therefore$ by theorem 2, $In(T) = (1, \sum_{i=1}^{n-1} 1, \sum_{i=1}^{n-1} 0) = (1, n-1, 0)$. ...□□□

Corollary 3.2:-

Let G be Lotus with blocks $G_1, G_2, \cdots, G_r$. Let u be the merging point of G_i, for $i = 1, 2, \ldots, r$. If $In(G_i) = (1, y_i, z_i)$ and $In((\widetilde{G_i})_u) = (0, y_i, z_i)$, where, $u \in U$

Then, $In(G) = (1, \sum_{i=1}^{r} y_i, \sum_{i=1}^{r} z_i)$.

Proof :-

Since every lotus is a cactus.
$\therefore$ result is true directly from Theorem 3. ...□□□

References

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