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Circular Reasoning In Geometrical Proofs: A Critical Discussion

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Abstract

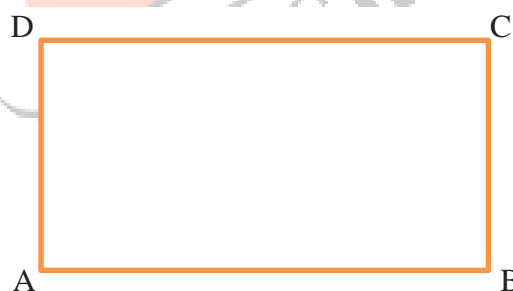
In geometry, a proof must be based on previously established definitions, axioms, and theorems. A common mistake made by students and sometimes even teachers is the use of circular reasoning, where a statement is proved by assuming the very result that is supposed to be established. This article discusses the theorem "Each angle of a rectangle is a right angle" and examines three proposed solutions. It explains why the first solution is valid and why the second and third solutions are invalid due to circular reasoning. A similar analysis is then made for the theorem "The sum of the three angles of a triangle is 180° ." Through these examples, the importance of logical order in mathematical proofs is emphasized.

Introduction:

Question:- Show that each angle of a rectangle is right angle.

First Solution: Let us recall what a rectangle is.

A rectangle is a parallelogram in which one angle is a right angle.



Let ABCD is a rectangle in which $\angle A = 90^\circ$,
We have to show that $\angle B = \angle C = \angle D = 90^\circ$

We have $AD \parallel BC$, AB is a transversal

So, $\angle A + \angle B = 180^\circ$

[Interior angles on the same side of the transversal]

But $\angle A = 90^\circ$

So, $\angle B = 90^\circ$

Now, $\angle C = \angle A$ and $\angle D = \angle B$

[Opposite angles of the parallelogram]

So, $\angle C = 90^\circ$ and $\angle D = 90^\circ$

Therefore, each angle of a rectangle is a right angle. \rightarrow showed

2. Question: Show that each angle of a rectangle is a right angle.

Second Solution:

Given:- ABCD is a rectangle.

We are to show that $\angle BAD = \angle ABC = \angle BCD = \angle ADC = 90^\circ$

Construction:- AC and BD diagonals are joined and O is their intersect point.

show: We know that diagonals of parallelogram bisect each other at their intersect point.

Therefore, $OA = OC$ and $OB = OD$

But $AC = BD$ [Diagonals of rectangle]

Therefore, $OA = OB = OC = OD$ ——— ①

We get from $\triangle AOB$

$OA = OB$ [From ①]

$\therefore \angle OBA = \angle OAB$

$\Rightarrow \angle OBA = \angle BAC$ ——— ②

Again, we get from

$\triangle BOC$, $OC = OB$ [From ①]

$\therefore \angle OBC = \angle OCB$

$\Rightarrow \angle OBC = \angle ACB$ ——— ③

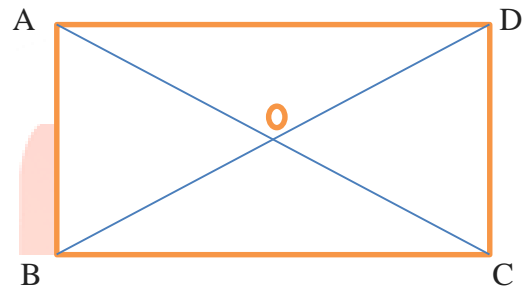
Now, ② + ③ \Rightarrow

$\angle OBA + \angle OBC = \angle BAC + \angle ACB$

$\Rightarrow \angle ABC = \angle BAC + \angle ACB$

But $\angle ABC + \angle BAC + \angle ACB = 180^\circ$

[Sum of three angles of a triangle is 180°]



Therefore, $\angle ABC = 90^\circ$

Similarly $\angle BAD = 90^\circ$

$\angle BCD = 90^\circ$

and $\angle ADC = 90^\circ$

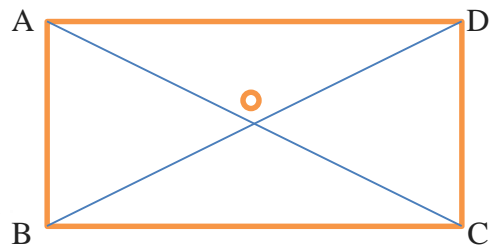
Therefore each angle of a rectangle is a right angle \rightarrow showed.

3. Question: Show that each angle of a rectangle is a right angle.

Third Solution:

Given:- ABCD is a rectangle. We are to show that
 $\angle BAD = \angle ABC = \angle BCD = \angle ADC = 90^\circ$

Construction:- AC and BD diagonals are joined.



show: With respect to
 $\triangle ABC$ and $\triangle BDC$

$AB = CD$

[Opposite sides of rectangle]

$AC = BD$

[Diagonals of rectangle]

and $BC = BC$

[common sides]

$\therefore \triangle ABC \cong \triangle BDC$

$\therefore \angle ABC = \angle BCD$ ----- ①

Now, $AB \parallel CD$, BC is their transversal.

$\therefore \angle ABC + \angle BCD = 180^\circ$

[Interior angles on the same side of the transversal]

$\Rightarrow \angle ABC + \angle ABC = 180^\circ$ [From ①]

$\Rightarrow 2 \angle ABC = 180^\circ$

$\Rightarrow \angle ABC = 90^\circ$

But $\angle ABC = \angle BCD$

Therefore, $\angle ABC = \angle BCD = 90^\circ$

Similarly $\angle BAD = \angle ADC = 90^\circ$

Now, we can write

$\angle BAD = \angle ABC = \angle BCD = \angle ADC = 90^\circ$

Therefore, each angle of a rectangle is a right angle.

showed

Discussion:

Consider the theorem:

"Show that each angle of a rectangle is a right angle."

Suppose three different solutions are presented for this theorem. It is generally accepted that the first solution is correct, whereas the second and third solutions are incorrect. At first glance, one may feel that the first solution is also wrong because it begins by taking $\angle A = 90^\circ$. Since the objective is to prove that the angles of a rectangle are right angles, it may appear that the proof is assuming what it is trying to establish.

However, a careful examination of the definition of a rectangle removes this doubt. By definition,

A rectangle is a parallelogram having one right angle.

Therefore, when we begin the proof by taking $\angle A = 90^\circ$, we are not assuming the conclusion. We are merely using the definition of a rectangle. Since the existence of one right angle is already part of the given information, it is perfectly legitimate to start with $\angle A = 90^\circ$ and then prove that the remaining three angles are also right angles.

Discussion of the Three Solutions:

Why the First Solution is Correct

Let ABCD be a rectangle.

By definition, ABCD is a parallelogram with one right angle. Therefore,

$$\angle A = 90^\circ.$$

Since opposite angles of a parallelogram are equal,

$$\angle C = \angle A = 90^\circ.$$

Also, adjacent angles of a parallelogram are supplementary.

Therefore,

$$\angle A + \angle B = 180^\circ$$

$$90^\circ + \angle B = 180^\circ$$

$$\angle B = 90^\circ.$$

Similarly,

$$\angle D = 90^\circ.$$

Hence all four angles of the rectangle are right angles.

This proof is valid because it uses only the definition of a rectangle and established properties of parallelograms.

Why the Second and Third Solutions are wrong

The second and third solutions attempt to use the theorem:

"The diagonals of a rectangle are equal."

However, the equality of the diagonals is itself a property that is derived after establishing that a rectangle has four right angles.

In other words, before proving the nature of the angles of a rectangle, one cannot use a theorem whose proof depends on those very angle properties.

The logical order is:

Definition of rectangle

→ Each angle is a right angle

→ Other properties of rectangles

→ Equality of diagonals

The reverse order is not logically acceptable.

A simple analogy may help.

A son comes from a father, but a father does not come from a son.

Similarly, if Statement A is used to prove Statement B, then Statement B cannot later be used to prove Statement A unless an independent proof is provided.

Therefore, the second and third solutions are examples of circular reasoning.

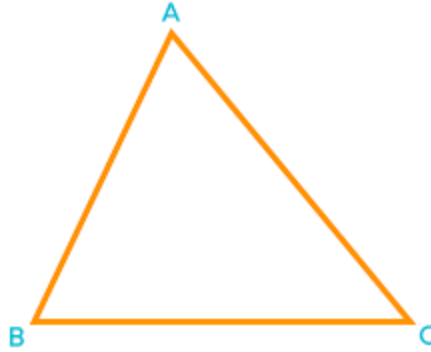
Unfortunately, some teachers may mistakenly regard the second and third solutions as correct. However, mathematical validity depends on logic, not on majority opinion. Since these solutions rely on a result that itself depends on the theorem being proved, they are not valid proofs.

Another Example of Circular Reasoning

Consider the theorem:

"Show that the sum of the three angles of a triangle is 180° ."

Given triangle ABC, we are required to prove



$$\angle BAC + \angle ABC + \angle ACB = 180^\circ.$$

Suppose someone argues as follows:

"We know that the sum of the three interior angles of a triangle is half the sum of the three exterior angles."

Since the sum of the three exterior angles is 360° ,

$$\begin{aligned} \angle BAC + \angle ABC + \angle ACB &= \frac{1}{2} \times 360^\circ \\ &= 180^\circ. \end{aligned}$$

At first sight, the argument appears correct. However, it is actually invalid.

The reason is that the theorem

"The sum of the three exterior angles of a triangle is 360° "

is generally proved using the theorem

"The sum of the three interior angles of a triangle is 180° ."

Therefore, if we use the exterior-angle theorem to prove the interior-angle theorem, we are using a result whose proof already depends on the statement being established.

The logical order is:

$$\text{Sum of interior angles} = 180^\circ \rightarrow \text{Sum of exterior angles} = 360^\circ$$

The reverse direction is not acceptable unless the exterior-angle theorem has been proved

independently.

Thus, this argument is another example of circular reasoning.

Conclusion:

Mathematical proofs must follow a correct logical sequence. A theorem may be proved only from definitions, axioms, and previously established results. The theorem "Each angle of a rectangle is a right angle" can validly begin with one angle being 90° because this fact is part of the definition of a rectangle. Therefore, the first solution is correct.

On the other hand, any proof that uses the equality of the diagonals of a rectangle before establishing the angle properties commits a logical error. Similarly, proving that the sum of the interior angles of a triangle is 180° by using a theorem that itself depends on that result is also invalid.

Such arguments are examples of circular reasoning. Circular reasoning occurs when a conclusion is used, directly or indirectly, to justify itself. Sound geometry requires avoiding this error and maintaining a proper logical order in every proof.

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