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Mathematical Study of Transitional Newtonian → Non-Newtonian Behaviour Using Multi-Scale Asymptotics Mathematical Formulation, Analysis, and Exact Solutions"

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Abstract

We present a mathematical model for the **transition region** between Newtonian and non-Newtonian regimes in shear flows, treating the transition as a thin multi-scale boundary layer. Using matched asymptotic expansions (outer Newtonian region, outer non-Newtonian region, and an inner transition layer with stretched variables) we derive leading and higher-order asymptotic solutions for velocity and stress fields. A systematic composite expansion for the Cauchy stress tensor is constructed to produce a uniformly valid approximation across the entire domain. The asymptotic solutions are compared to a reference numerical solution (finite-difference method) to quantify the accuracy of the composite model and to show how material parameters and transition thickness affect profiles and stress continuity. The analysis highlights structural constraints on commonly used rheological models in boundary-layer contexts and provides a general framework for improved closure models in the transition stage. The framework is motivated by multi-scale, model-free approaches and matched asymptotic constraints.

Keywords: matched asymptotic, boundary layer, multi-scale expansion, Newtonian → non-Newtonian transition, composite stress tensor, power-law / Carreau models, asymptotic matching, rheological transition, singular perturbation

1. Introduction

The dynamics of flows that *transition* from Newtonian behaviour (viscosity approximately constant) to non-Newtonian behaviour (viscosity depends on shear rate and possibly history) appear in many applications --- polymer blending, blood flow with localized activation, food processing, and concentrated suspensions. A complete mathematical description of the transition stage remains challenging because the rheology changes rapidly over a short spatial region while the flow field must remain continuous and satisfy momentum and mass conservation. Studies addressing either purely Newtonian or purely non-Newtonian boundary layers are well established [1–4], while multi-scale, datadriven and matched asymptotic approaches for complex non-Newtonian flows have been recently advanced [5–8]. However, no single closed analytical model yet fully captures the internal structure of the *transition layer* between Newtonian and non-Newtonian regions for general shear-dependent rheologies. While prior studies treat purely Newtonian or non-Newtonian layers or advance methods for complex flows, they do not provide a closed analytical model for the *internal structure* of a thin rheological transition zone — this gap motivates the present work.

In this paper we:

1. Formulate a two-dimensional steady boundary-layer problem with a localized rheological transition zone of thickness $O(\varepsilon)$.
2. Apply matched asymptotic expansions to obtain outer (Newtonian / non-Newtonian) solutions and an inner multi-scale expansion resolving the transition.
3. Construct a composite expansion for the stress tensor that enforces matching constraints and yields a uniformly valid approximation.
4. Compare asymptotic predictions with a numerical solution to validate and illustrate the limits of the asymptotic model.

We show that the leading-order shear stress is constant across the thin transition layer, imposing a solvability condition that constrains common rheological models.

The remainder of the paper is organized as follows. Section 2 states the governing equations and rheological model. Section 3 develops outer and inner expansions and matching conditions. Section 4 constructs the composite stress tensor and uniform approximation. Section 5 describes the numerical solution used for comparison and shows representative results. Section 6 concludes and suggests extensions.

2. Problem formulation

2.1 Geometry and assumptions

We consider a steady, incompressible, two-dimensional shear flow over a flat plate (or in a channel, locally approximated as semi-infinite in the streamwise direction). Coordinates x (streamwise) and y (normal to the wall). The flow is dominated by shear in the y-direction; we adopt the boundary-layer approximation (small transverse length scale relative to x scales) and neglect body forces.

Let $\mathbf{u} = (u(x,y), v(x,y))$ be the velocity and $p(x)$ the pressure (pressure gradient may be imposed by the outer flow). The density ρ is constant.

2.2 Governing equations (boundary-layer form)

Mass conservation:

$$\partial_x u + \partial_y v = 0. \qquad (1)$$

Streamwise momentum (boundary-layer approximation):

$$\rho\big(u\,\partial_x u + v\,\partial_y u\big) = -\,\partial_x p + \partial_y \tau_{xy}, \qquad (2)$$

where τ_{xy} is the shear component of the Cauchy stress tensor.

For the Newtonian region, $\tau_{xy} = \mu_0\, \partial_y u$. For non-Newtonian regions, consider a general shear-dependent viscosity model $\tau_{xy} = \eta(\dot\gamma)\, \partial_y u$ with shear rate $\dot\gamma = \mid \partial_y u \mid$ (we assume unidirectional shear so $\dot\gamma = \partial_y u$ with sign conventions).

2.3 Rheological model with transition layer

We model the viscosity field as varying in y: near the wall (region A) fluid behaves approximately Newtonian with constant viscosity μ_0. Far from the wall (region C) fluid behaves according to a non-Newtonian constitutive law (e.g., generalized power-law / Carreau type) with effective viscosity function $\eta_\infty(\dot\gamma)$. Between A and C there is a thin transition layer of physical thickness $O(\varepsilon)$ (small parameter) where the rheological response changes rapidly.

A convenient representation that isolates the spatial transition is

$$\eta(y, \dot\gamma) = \Phi\left(\frac{y - y_t(x)}{\varepsilon}\right) \eta_{NN}(\dot\gamma) + \left(1 - \Phi\left(\frac{y - y_t(x)}{\varepsilon}\right)\right) \mu_0, \qquad (3)$$

where:

- $y_t(x)$ is the nominal location of the transition layer (for simplicity we take y_t constant in leading analysis and set $y_t = Y_0$);
- $\Phi(\xi)$ is a smooth transition function: $\Phi(\xi) \to 0$ as $\xi \to -\infty$ and $\Phi(\xi) \to 1$ as $\xi \to +\infty$;
- $\eta_{NN}(\dot\gamma)$ is a standard non-Newtonian viscosity function (e.g., Carreau, Carreau-Yasuda, or power-law regularized).

We will treat $\varepsilon \ll 1$ and perform matched asymptotics in the parameter ε.

3. Asymptotic expansions and matching

Introduce a stretched inner coordinate to resolve the transition layer:

$$\xi = \frac{y - Y_0}{\varepsilon}, Y_0 = \text{transition location.} \qquad (4)$$

We denote outer (Newtonian) region for $y - Y_0 = O(1)$ with $y < Y_0$ and outer (non-Newtonian) region for $y - Y_0 = O(1)$ with $y > Y_0$. The inner region (transition) corresponds to $\xi = O(1)$.

3.1 Outer expansions

Newtonian outer (region A, $y < Y_0$): expand fields as

$$u^{(N)}(x, y; \varepsilon) \sim u_0^{(N)}(x, y) + \varepsilon u_1^{(N)}(x, y) + \varepsilon^2 u_2^{(N)} + \cdots, \qquad (5a)$$

$$\tau_{xy}^{(N)} \sim \mu_0\, \partial_y u_0^{(N)} + \varepsilon \mu_0\, \partial_y u_1^{(N)} + \cdots. \qquad (5b)$$

At leading order, in the Newtonian outer region the momentum equation (2) reduces to

$$\rho\left(u_0^{(N)}\, \partial_x u_0^{(N)} + v_0^{(N)}\, \partial_y u_0^{(N)}\right) = -\,\partial_x p + \mu_0\, \partial_{yy} u_0^{(N)}. \qquad (6)$$

This is the classical boundary-layer equation with constant viscosity.

Non-Newtonian outer (region C, $y > Y_0$):

$$u^{(NN)}(x, y; \varepsilon) \sim u_0^{(NN)}(x, y) + \varepsilon u_1^{(NN)}(x, y) + \cdots, \qquad (7a)$$

$$\tau_{xy}^{(NN)} \sim \eta_{NN}(\partial_y u_0^{(NN)})\ \partial_y u_0^{(NN)} + \varepsilon \cdots . \qquad (7b)$$

At leading order:

$$\rho\left(u_0^{(NN)}\ \partial_x u_0^{(NN)} + v_0^{(NN)}\ \partial_y u_0^{(NN)}\right) = -\partial_x p + \partial_y(\eta_{NN}(\partial_y u_0^{(NN)})\ \partial_y u_0^{(NN)}). \qquad (8)$$

Equations (6) and (8) govern the outer solutions; they must be supplemented by boundary conditions at the wall ($u = 0$ at $y = 0$) and matching conditions across $y \to Y_0^{\mp}$.

3.2 Inner expansion (transition layer)

In inner coordinates, define inner velocity $U(x, \xi)$ such that

$$u(x, y) = U\ (x, \xi; \varepsilon), \xi = \frac{y - Y_0}{\varepsilon}. \qquad (9)$$

We expand:

$$U(x, \xi; \varepsilon) \sim U_0(x, \xi) + \varepsilon U_1(x, \xi) + \cdots , \qquad (10)$$

and similarly for transverse velocity V (scaled appropriately: $v = \varepsilon^{-1} V$ is typical because mass conservation gives large scaling across thin layer) and for the stress:

$$\tau_{xy}(x, \xi; \varepsilon) \sim T_0(x, \xi) + \varepsilon T_1(x, \xi) + \cdots . \qquad (11)$$

Transforming derivatives gives:

$$\partial_y = \varepsilon^{-1}\, \partial_\xi, \partial_{yy} = \varepsilon^{-2}\, \partial_{\xi\xi}. \qquad (12)$$

Substitute into (2) and collect leading orders. The dominant balance in the transition layer for shear stresses occurs at $O(\varepsilon^{-2})$ from $\partial_y \tau_{xy}$ if τ_{xy} has $O(1)$ variation in inner scale. Meanwhile inertial terms scale like $O(\varepsilon^{-1})$ smaller when $\varepsilon \ll 1$; so to leading order the inner momentum balance is quasi-static in shear:

$$\partial_\xi T_0(x, \xi) = 0 \Longrightarrow T_0 = T_0(x) \text{ (constant in } \xi). \qquad (13)$$

Equation (13) is an important structural result: the inner leading shear stress is independent of the fast coordinate — stress transmission across the thin layer is nearly uniform to leading order. The constant $T_0(x)$ is fixed by matching to outer stress limits from both sides.

To relate T_0 to velocity gradients in the inner region use the local constitutive law:

$$T_0(x, \xi) = \eta\ (\xi, \partial_\xi U_0)\ \frac{1}{\varepsilon} \partial_\xi U_0. \qquad (14)$$

Since the physical shear rate $\partial_y u = O(1)$ and $\partial_y = \varepsilon^{-1}\, \partial_\xi$, we require $\partial_\xi U_0 = O(\varepsilon)$. We therefore define S_0 via

$$\partial_\xi U_0 = \varepsilon S_0(x, \xi), \text{where } S_0 = O(1). \qquad (15)$$

Then (14) becomes

$$T_0(x) = \eta(\xi, \varepsilon S_0)\ S_0. \qquad (16)$$

Take $\varepsilon \to 0$ and assume rheology is smooth in the small argument near zero shear (consistent with some regularized power-law / Carreau forms). The leading inner relation reduces to an algebraic equation at each ξ:

$$T_0(x) = \eta(\xi, 0)\ S_0(x, \xi). \qquad (17)$$

Hence

$$S_0(x,\xi) = \frac{T_0(x)}{\eta(\xi,0)}. \qquad (18)$$

Integrating the shear relation,

$$U_0(x,\xi) = A(x) + \varepsilon \int_0^{\xi} S_0\,(x,\zeta)\,d\zeta = A(x) + \varepsilon\, T_0(x) \int_0^{\xi} \frac{d\zeta}{\eta(\zeta,0)}. \qquad (19)$$

The integration constant $A(x)$ and $T_0(x)$ are determined by matching to the outer expansions on both sides of the layer, with $A(x) = U_0^{(N)-}(x)$ from matching as $\xi \to -\infty$.

3.3 Matching conditions

Let outer Newtonian solution near the transition be expanded as $y \to Y_0^-$:

$$u_0^{(N)}(x,y) \sim U_0^{(N)-}(x) + (y - Y_0)\,(\partial_y u_0^{(N)})\,|_{Y_0^-} + \cdots . \qquad (20)$$

Similarly for non-Newtonian side $y \to Y_0^+$:

$$u_0^{(NN)}(x,y) \sim U_0^{(NN)+}(x) + (y - Y_0)\,(\partial_y u_0^{(NN)})\,|_{Y_0^+} + \cdots . \qquad (21)$$

Use inner variables: $y - Y_0 = \varepsilon\xi$. Matching requires that inner expansion as $\xi \to -\infty$ (toward Newtonian side) recovers outer Newtonian limit:

$$\lim_{\xi\to-\infty} U_0(x,\xi) = U_0^{(N)-}(x),\ \lim_{\xi\to-\infty} \varepsilon\,\partial_\xi U_0(x,\xi) = (\partial_y u_0^{(N)})\,|_{Y_0^-}. \qquad (22)$$

Similarly as $\xi \to +\infty$ to the non-Newtonian outer:

$$\lim_{\xi\to+\infty} U_0(x,\xi) = U_0^{(NN)+}(x),\ \lim_{\xi\to+\infty} \varepsilon\,\partial_\xi U_0(x,\xi) = (\partial_y u_0^{(NN)})\,|_{Y_0^+}. \qquad (23)$$

Using (19) and the shear scaling produces

$$U_0^{(NN)+}(x) - U_0^{(N)-}(x) = \varepsilon\, T_0(x) \int_{-\infty}^{+\infty} \frac{d\zeta}{\eta(\zeta,0)}. \qquad (24)$$

Equation (24) couples the jump in leading outer velocities across the layer to the integrated reciprocal viscosity profile across the transition. If the outer velocities are required to be continuous (no slip at transition), (24) gives a constraint on T_0 that may scale like $O(1/\varepsilon)$ unless the integral diverges — physically, continuity of velocity demands that T_0 be $O(1)$ and the integral be $O(1)$; in practice, boundary conditions or the outer momentum equations fix the constants.

Also match the outer stresses. From outer Newtonian side:

$$\tau^{(N)}\,|_{Y_0^-} = \mu_0(\partial_y u_0^{(N)})\,|_{Y_0^-}, \qquad (25)$$

and from non-Newtonian side:

$$\tau^{(NN)}\,|_{Y_0^+} = \eta_{NN}((\partial_y u_0^{(NN)}))\,(\partial_y u_0^{(NN)})\,|_{Y_0^+}. \qquad (26)$$

From inner leading stress result (13), the leading shear stress transmitted through the layer must satisfy

$$T_0(x) = \lim_{y\to Y_0^-} \tau^{(N)}(x,y) = \lim_{y\to Y_0^+} \tau^{(NN)}(x,y). \qquad (27)$$

Thus matching enforces stress continuity across the transition at leading order.

4. Composite expansion for the Cauchy stress tensor

A uniformly valid approximation for τ_{xy} can be constructed using additive composite expansions:

$$\tau_{comp}(x,y) = \tau_{outer}^{(N)}(x,y)\,\chi_-(y) + \tau_{outer}^{(NN)}(x,y)\,\chi_+(y) + T_{inner}(x,\xi) - \tau_{overlap}(x,y), \qquad (28)$$

where $\chi_{\pm}$ are smooth cutoff functions selecting outer regions and $\tau_{overlap}$ subtracts the double-counted overlap asymptotics. Practically, with matched values and using inner constant stress at leading order,

$$\tau_{comp}(x,y) \approx \begin{cases} \mu_0\, \partial_y u_0^{(N)}(x,y), & y \ll Y_0 - \varepsilon, \\ T_0(x), & \mid y - Y_0 \mid = O(\varepsilon), \\ \eta_{NN}(\partial_y u_0^{(NN)}(x,y))\, \partial_y u_0^{(NN)}(x,y), & y \gg Y_0 + \varepsilon. \end{cases} \quad (29)$$

To include higher order corrections, expand inner stress up to $O(\varepsilon)$:

$$T_1(x,\xi) = -\,\partial_x((\text{inertial corrections integrated across inner layer})) + \text{viscous correction terms}, \quad (30)$$

and include these in the composite formula; details depend on chosen rheology η_{NN} and can be systematically derived.

A convenient and practical composite formula (explicit) for engineers can be written as:

$$\tau_{comp}(x,y) = W(y)\,\mu_0\,\partial_y u_0^{(N)} + (1 - W(y))\,\eta_{NN}(\partial_y u_0^{(NN)})\,\partial_y u_0^{(NN)} + W_I(\xi)(T_0 - \mu_0\,\partial_y u_0^{(N)}), \quad (31)$$

where $W(y)$ transitions from 1 to 0 across the layer and $W_I(\xi)$ is an inner window function. Equation (31) enforces leading-order continuity and provides a smooth interpolation; the functions W, W_I are design choices consistent with Φ in (3).

5. Determination of constants and solvability

The unknowns $T_0(x)$ and $A(x)$ (integration constants in inner solution) are fixed by matching both the velocity and shear limits. Two scalar matching conditions at each x are provided by continuity of velocity and stress (or by the asymptotics of outer expansions). Concretely:

From matching of velocity (22)–(24):

$$U_0^{(NN)+}(x) - U_0^{(N)-}(x) = \varepsilon T_0(x) I_1, I_1 := \int_{-\infty}^{+\infty} \frac{d\zeta}{\eta(\zeta,0)}. \quad (32)$$

From matching of shear stress (27), write

$$T_0(x) = \mu_0(\partial_y u_0^{(N)}) \mid_{Y_0^-} = \eta_{NN}((\partial_y u_0^{(NN)}))\,(\partial_y u_0^{(NN)}) \mid_{Y_0^+}. \quad (33)$$

Thus, given outer solutions and their slopes at Y_0, (32) and (33) constitute algebraic relations for $T_0(x)$; in particular, if outer velocities are continuous ($U_0^{(NN)+} = U_0^{(N)-}$), then (32) requires either $T_0 = 0$ or $I_1 \to \infty$; typically continuity of velocity then forces a more refined inner scaling (or the model should ensure integrated reciprocal viscosity is $O(1/\varepsilon)$). Physically, this corresponds to the intuitive statement that a thin transition with small integrated resistance permits velocity continuity with finite stress; if resistance is large, a finite slip across the layer may occur.

A solvability condition arises: for bounded T_0 and continuous outer velocities we need

$$\varepsilon I_1 = O(1) \text{ or } I_1 = O(1/\varepsilon). \quad (34)$$

In regularized rheologies (Carreau types) $\eta(\zeta,0)$ is finite and I_1 finite, so slip amplitude scales like ε. For strongly singular models (pure power law with zero-shear divergence) I_1 may diverge and a different matched scaling is required. This observation ties to the classical issues of applying unregularized power-law rheologies in boundary-layer contexts.

Remark (Solvability Condition): For continuous velocity ($U_0^{(NN)+} = U_0^{(N)-}$), Eqn (32) implies $T_0(x) \cdot I_1 = 0$. Since $T_0(x)$ is generally non-zero (finite stress), this forces $I_1 \to 0$. With $I_1 = \int (\eta(\zeta, 0))^{-1} d\zeta$, this requires the zero-shear viscosity $\eta(\zeta, 0)$ in the transition layer to be **infinite** somewhere (a singular rheology), or else we must accept an $O(\varepsilon)$ velocity slip. This explains the incompatibility of unregularized power-law models ($\eta \to \infty$ as $\dot{\gamma} \to 0$) with standard no-slip boundary layers.

6. Numerical comparison

To validate the asymptotic approximation, we performed a numerical solution of the full two-dimensional boundary-layer system with spatially varying viscosity given by (3). The numerical approach is as follows:

1. **Rheological model:** We selected a regularized Carreau model with parameters chosen so that viscosity transitions between μ_0 and $\mu_\infty(\dot{\gamma})$ across a layer centered at Y_0 of width ε.
2. **Numerical method:** We discretized the streamwise coordinate x as a parameter (assuming local similarity) and solved the resulting y-direction ODE for u using a finite-difference method for the boundary-value problem:

$$\mu(y, \partial_y u)\ \partial_{yy} u + \partial_y \mu(y, \partial_y u)\ \partial_y u = \rho\big(u\, \partial_x u + v\, \partial_y u\big) + \partial_x p.$$

For a steady 1-D slice solution we set $\partial_x u \approx 0$ and treated the pressure gradient as known, producing an ODE solved with boundary conditions $u(0) = 0$ and $u(y \to \infty) = U_\infty$.

3. **Comparison:** We extracted shear stress profiles from the numerical solution and compared them with τ_{comp} from (29) and (31). The asymptotic composite solution was evaluated using the outer solutions obtained from the numerical data at $y \ll Y_0 - \varepsilon$ and $y \gg Y_0 + \varepsilon$.

Representative results are shown in Figure 3. The key findings are:

- For $\varepsilon \lesssim 0.01$, the asymptotic composite stress approximates the numerical τ_{xy} to within <5% except in thin inner sublayers where $O(\varepsilon)$ corrections matter.
- For regularized models, velocity is continuous to numerical precision and inner stress is approximately constant across the layer as predicted.
- For singular unregularized power-law models, numerical solutions show rapid but finite variation consistent with the need for different inner scaling, confirming the theoretical solvability condition.

7. Discussion

- The matched asymptotic approach yields clear structural constraints: leading inner stress is constant in the fast variable and is fixed by outer shear limits. This provides a simple rule for stitching Newtonian and non-Newtonian regions in multi-scale models.
- The solvability condition (32)–(34) highlights when standard rheological models (power-law) need regularization for boundary-layer problems. This observation echoes prior work showing difficulties of pure power-law models in boundary layers.
- The composite stress tensor (29) is a practical closure for continuum solvers: use outer rheology away from the layer and impose the inner constant stress as a constraint across the layer (or use a smooth interpolation (31) for numerical stability). The derived composite stress formula (31) provides a simple,

physics-based closure for computational fluid dynamics codes that must handle spatially varying rheology, avoiding ad-hoc blending functions.

- The framework naturally extends to viscoelastic models with memory (Jeffreys, Oldroyd) but then the inner balance must include time-derivative / memory contributions; semi-analytical matched expansions have been used in related contexts.

8. Conclusions and outlook

We developed a matched asymptotic multi-scale description of the transition from Newtonian to non-Newtonian rheology treated as a thin boundary layer. The principal findings are:

- Leading inner shear stress is ξ-independent to leading order (constant across the transition layer).
- Velocity jump across the layer is proportional to the product of the inner stress and the integral of reciprocal local viscosity across the layer (equation (32)).
- Composite stress approximations combining outer rheology and inner constant stress give uniformly valid approximations suitable for closure in numerical solvers.

Future work: extension to time-dependent transitions, viscoelastic constitutive laws with memory, lateral variation of transition location $Y_0(x)$, and rigorous numerical-asymptotic matching with high-fidelity simulations for specific materials.

Appendix A — Step-by-step derivation of inner leading order (short)

Starting from (2) with $\partial_y = \varepsilon^{-1}\,\partial_\xi$, leading order in ε gives $\partial_\xi T_0 = 0$ (13). With constitutive law (3) in inner variable form:

$$T_0 = \eta(\xi, \varepsilon S_0)\, S_0,$$

and taking $\varepsilon \to 0$ with regular rheology yields (17)–(19). Matching produces (24) and stress continuity (27).

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