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Fixed Point Theorem With Rational Expansive Mappings In Complex Valued Partial Metric Space

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Abstract

This paper establishes a common fixed-point theorem in complex valued partial metric spaces using rational expansive mappings and newly defined implicit relations. Under the framework of occasionally weakly compatible mappings, the existence and uniqueness of a common fixed point for four self-mappings are proved. Our result is generalization of some of the results existing in the literature.

Keywords: Common fixed point, Complex valued partial metric space, Rational expansive mapping, Occasionally Weakly Compatible (OWC) mappings, Implicit relation

1 Introduction

Over the past century, fixed point theory a foundational field of nonlinear analysis has undergone substantial development. The classical Banach Contraction Principle [5], which ensures the existence and uniqueness of fixed points in complete metric spaces, where this topic first emerged. Numerous generalizations and applications in mathematics and the applied sciences were made possible by this outcome. Later, the idea of metric spaces was expanded in a number of ways. For example, S. G. Matthews [19] introduced partial metric spaces, which offer a helpful framework in domain theory and computer science and do not require a point's self-distance to be zero.

The complex valued metric spaces [4], in which the distance function takes values in the complex plane, were created as a result of further generalization. When these concepts were combined, complex valued partial metric spaces became a potent framework for studying more general fixed-point problems. To widen on traditional fixed-point results, researchers have added a variety of contractive conditions over time which includes weak contractions, rational contractions, and expansive mappings. Implicit relations in particular have become more significant as they can combine several contraction conditions into a single framework.

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In this context, new implicit relations for rational expansive mappings in complex valued partial metric spaces are introduced in this work. A common fixed-point theorem for four self-mappings is established by using the concept of occasionally weakly compatible mappings. The obtained results contribute to the development of generalized metric space theory by guaranteeing the existence and uniqueness of the common fixed point and generalizing a number of previous findings in the literature.

The rest of the paper is structured as follows. Section 2 recalls preliminaries on complex valued partial metric spaces. Section 3 presents our main results with detailed proof and two illustrative examples to support our main theorem. Section 4 concludes the article.

2 Preliminaries and Notations

The following are useful in our main results which are due to [9].

Definition 2.1 (Complex Valued Partial Metric Space). *Let* X *be a non-empty set. A function*

$$p: X \times X \to \mathbb{C}$$

is called a complex valued partial metric if for all $x, y, z \in X$, *it satisfies:*

- $0 \preceq p(x,x) \preceq p(x,y)$
- $p(x,y) = p(y,x)$
- $x = y \iff p(x,x) = p(x,y) = p(y,y)$
- $p(x,y) \preceq p(x,z) + p(z,y) - p(z,z)$

Then (X,p) *is called a complex valued partial metric space.*

Definition 2.2. (Convergence). Let (X,p) be a complex valued partial metric space. A sequence $\{x_n\}$ in X is said to converge to a point $x \in X$ if

$$\lim_{n\to\infty} p(x_n, x) = p(x,x)$$

Definition 2.3. (Cauchy sequence). *A sequence* $\{x_n\}$*in* (X, p)*is called a Cauchy sequence if the limit*

$$\lim_{n,m\to\infty} p(x_n, x_m)$$

exists (is finite in $\mathbb{C}$*).*

Definition 2.4. (Completeness). *Let* (X,p) *be a complex valued partial metric space. It is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges to some point* $x \in X$ *such that*

3 Main Results

Now we prove our main theorem.

New implicit relations for rational expansive condition:

Let φ_6 be the set of continuous functions $\Phi: (\mathbb{R}^+)^6 \to \mathbb{R}$ non decreasing in the first argument, non-increasing in the 2nd ,3rd ,4th, 5th arguments such that

(i) $\Phi(u, v, v, v, v, w) \leq 0 \Rightarrow |u| \geq \beta|v| \; for \; \beta > 1$

(ii) $\Phi(u, u, 1,1,1, u) \leq 0 \Rightarrow u \geq 1$

Theorem: *Let (X, p) be a complete complex valued partial metric space. Let $A, B, S, T: X \to X$ be four self-mappings with $\{A, S\}$ and $\{B, T\}$ occasionally weakly compitable. Suppose there exists $q > 1$ and $\Phi \epsilon \Phi_6$ such that for all $x_0, y_0 \epsilon X$*

$\Phi(|p(Sx, Ty)|, |p(Ax, By)|, |p(Sx, Ax)|, |p(By, Ty)|, \frac{|p(Ax,Sx)|.|p(By,Ty)|}{1+|p(Ax,Sx)|+|p(By,Ty)|}, |p(Ax, Ty)| + |p(By, Sx)|) \leq 0 \ldots$ *(3.1)*

Then A, B, S, T have a unique common fixed point in X.

Proof: Since the pair $\{A, S\}$ is occasionally weakly compatible, by definition there exists a point $x_0 \in X$ such that:

$$Ax_0 = Sx_0$$

Let the common value be u. Then

$$u = Ax_0 = Sx_0$$

Similarly, since the pair $\{B, T\}$ is occasionally weakly compatible, there exists a point $y_0 \in X$ such that:

$$By_0 = Ty_0$$

Let the common value be v. Then

$$v = By_0 = Ty_0$$

We have to prove u = v

Assume (contradiction) that $u \neq v$. We have to show this as contradiction.

Substituting $x = x_0$ and $y = y_0$ into the implicit inequality (3.1). Since $Sx_0 = Ax_0 = u$ and $Ty_0 = By_0 = v$

The fifth term in (3.1) becomes:

$$\frac{|\, p(Ax_0, Sx_0) \,| \cdot |\, p(By_0, Ty_0) \,|}{1 + |\, p(Ax_0, Sx_0) \,| + |\, p(By_0, Ty_0) \,|} = \frac{|\, p(u,u) \,| \cdot |\, p(v,v) \,|}{1 + |\, p(u,u) \,| + |\, p(v,v) \,|}$$

The sixth term becomes:

$$|\, p(Ax_0, Ty_0) \,| + |\, p(By_0, Sx_0) \,| = |\, p(u,v) \,| + |\, p(v,u) \,| = 2 \,|\, p(u,v) \,|$$

Thus, inequality (3.1) reduces to:

$$\Phi(|\, p(u,v) \,|, \; |\, p(u,v) \,|, \; |\, p(u,u) \,|, \; |\, p(v,v) \,|, \; \frac{|\, p(u,u) \,| \cdot |\, p(v,v) \,|}{1 + |\, p(u,u) \,| + |\, p(v,v) \,|}, 2 \,|\, p(u,v) \,|) \leq 0 \quad (3.2)$$

Now, by the axioms of a complex valued partial metric space. For any points $u, v \in X$, we have:

$$0 \precsim p(u,u) \precsim p(u,v) \text{ and } 0 \precsim p(v,v) \precsim p(u,v)$$

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Taking moduli (since the partial order $\preceq$ on $\mathbb{C}$ is defined component-wise, and all quantities are non-negative real numbers after taking moduli), we obtain:

$$|p(u,u)| \le |p(u,v)| \text{ and } |p(v,v)| \le |p(u,v)| \qquad (3.3)$$

Consider the fifth term. Observe that for any non-negative real numbers α, β:

$$\frac{\alpha\beta}{1+\alpha+\beta} \le \min(\alpha, \beta)$$

This is a standard inequality. Applying it with $\alpha = |p(u,u)|$ and $\beta = |p(v,v)|$, we get:

$$\frac{|p(u,u)| \cdot |p(v,v)|}{1+|p(u,u)|+|p(v,v)|} \le \min(|p(u,u)|,\ |p(v,v)|) \le |p(u,v)| \qquad (3.4)$$

where the last inequality comes from (3.3).
Therefore, we have the following bounds:

$$|p(u,u)| \le |p(u,v)|, |p(v,v)| \le |p(u,v)|, \frac{|p(u,u)| \cdot |p(v,v)|}{1+|p(u,u)|+|p(v,v)|} \le |p(u,v)|$$

Since Φ is non-increasing in its second, third, fourth, and fifth arguments, replacing these arguments with larger values can only make Φ smaller (or equal). More precisely, if $a \le a'$, then $\Phi(\cdots, a, \cdots) \ge \Phi(\cdots, a', \cdots)$ because Φ decreases when arguments increase.
Hence, from (3.2) and using the bounds above, we get:

$$\Phi(|p(u,v)|,\ |p(u,v)|,\ |p(u,v)|,\ |p(u,v)|,\ |p(u,v)|,\ 2|p(u,v)|) \le 0 \qquad (3.5)$$

Let us take:

$$\alpha = |p(u,v)|$$

Then inequality (3.5) becomes:

$$\Phi(\alpha,\ \alpha,\ \alpha,\ \alpha,\ \alpha,\ 2\alpha) \le 0 \qquad (3.6)$$

Now, observe that $\Phi(\alpha, \alpha, \alpha, \alpha, \alpha, 2\alpha)$ is of the form $\Phi(u, v, v, v, v, w)$ with $u = \alpha$ and $v = \alpha$. Therefore, we can apply the implicit condition with $\beta = q > 1$.
That is:

$$\Phi(u, v, v, v, v, w) \le 0 \implies |u| \ge \beta |v| \text{ for } \beta > 1$$

Substituting $u = \alpha$ and $v = \alpha$, we get:

$$\Phi(\alpha, \alpha, \alpha, \alpha, \alpha, 2\alpha) \le 0 \implies \alpha \ge q \cdot \alpha$$

Since $q > 1$, the inequality $\alpha \ge q\alpha$ implies:

$$\alpha - q\alpha \ge 0 \implies \alpha(1-q) \ge 0$$

Because $1 - q < 0$ (as $q > 1$), the product $\alpha(1-q) \ge 0$ forces $\alpha \le 0$. But $\alpha = |p(u,v)| \ge 0$. Hence:

$$\alpha = 0 \text{ that is } |p(u,v)| = 0$$

Since the modulus of a complex number is zero if and only if the complex number itself is zero, we have:

$$p(u,v) = 0$$

Now, from the partial metric definition, $p(u,v) = 0$ together with $p(u,u) \preceq p(u,v)$ and $p(v,v) \preceq p(u,v)$ implies:

$$p(u,u) = p(u,v) = p(v,v) = 0$$

By the partial metric definition, this forces:

$$u = v$$

This contradicts our assumption that $u \neq v$. Therefore, we must have:

$$u = v$$

Let us denote this common value by w. Thus:

$$w = Ax_0 = Sx_0 = By_0 = Ty_0 \qquad (3.7)$$

Now proving Uniqueness of the Coincidence Point for Each Pair

We prove that w is the unique coincidence point of A and S. Suppose there exists another point $z \in X$ such that $Az = Sz$.
Substitute $x = z$ and $y = y_0$ into inequality (3.1). Using $Sz = Az$, $Ty_0 = By_0 = w$, we obtain:

$$\Phi(|\, p(Az,w)\,|,\ |\, p(Az,w)\,|,\ |\, p(Az,Az)\,|,\ |\, p(w,w)\,|,\ \frac{|\, p(Az,Az)\,|\cdot|\, p(w,w)\,|}{1+|\, p(Az,Az)\,|+|\, p(w,w)\,|}, 2\,|\, p(Az,w)\,|) \leq 0$$

From the partial metric definition, $|\, p(Az,Az)\,| \leq |\, p(Az,w)\,|$ and $|\, p(w,w)\,| \leq |\, p(Az,w)\,|$. Also, $\frac{|p(Az,Az)|\cdot|p(w,w)|}{1+|p(Az,Az)|+|p(w,w)|} \leq |\, p(Az,w)\,|$.
Since Φ is non-increasing in arguments 2 to 5, replacing these arguments with the larger value $|\, p(Az,w)\,|$ which gives:

$$\Phi(|\, p(Az,w)\,|,\ |\, p(Az,w)\,|,\ |\, p(Az,w)\,|,\ |\, p(Az,w)\,|,\ |\, p(Az,w)\,|, 2\,|\, p(Az,w)\,|) \leq 0$$

Let $\beta = |\, p(Az,w)\,|$. Then $\Phi(\beta,\beta,\beta,\beta,\beta,2\beta) \leq 0$.
By condition (i), $\beta \geq q\beta$ with $q > 1$, which forces $\beta = 0$. Thus $p(Az,w) = 0$, so $Az = w$.
Since $Az = Sz$, we have $Sz = w$.
Now substitute $x = x_0$ and $y = z$ into inequality (3.1). Using $Sx_0 = Ax_0 = w$ and $Tz = Bz = z$, we get:

$$\Phi(|\, p(w,z)\,|,\ |\, p(w,z)\,|,\ |\, p(w,w)\,|,\ |\, p(z,z)\,|, \frac{|\, p(w,w)\,|\cdot|\, p(z,z)\,|}{1+|\, p(w,w)\,|+|\, p(z,z)\,|}, 2\,|\, p(w,z)\,|) \leq 0$$

From the partial metric definition, $|\, p(w,w)\,| \leq |\, p(w,z)\,|$ and $|\, p(z,z)\,| \leq |\, p(w,z)\,|$.
Also, $\frac{|p(w,w)|\cdot|p(z,z)|}{1+|p(w,w)|+|p(z,z)|} \leq |\, p(w,z)\,|$.
Since Φ is non-increasing in arguments 2 to 5, replacing these arguments with the larger value $|\, p(w,z)\,|$ we have:

$$\Phi(|\, p(w,z)\,|,\ |\, p(w,z)\,|,\ |\, p(w,z)\,|,\ |\, p(w,z)\,|,\ |\, p(w,z)\,|, 2\,|\, p(w,z)\,|) \leq 0$$

Let $\gamma = |\, p(w,z)\,|$. Then $\Phi(\gamma,\gamma,\gamma,\gamma,\gamma,2\gamma) \leq 0$. By condition (i), $\gamma \geq q\gamma$ with $q > 1$, which forces $\gamma = 0$.
Thus $p(w,z) = 0$, so $w = z$.
Therefore, w is the unique coincidence point of A and S.
Similarly, w is also the unique coincidence point of B and T.

We now prove that w is a fixed point of all four mappings.
Since $\{A, S\}$ is occasionally weakly compatible, there exists a coincidence point. We already have $Ax_0 = Sx_0 = w$. By the owc property at this point:

$$ASx_0 = SAx_0 \implies Aw = Sw \qquad (3.8)$$

Now, substitute $x = w$ and $y = y_0$ into inequality (3.1). We have $Sw = Aw$ (from 3.8),$Ty_0 = By_0 = w$ Substituting into (3.1):

$$\Phi(|p(Aw,w)|,\ |p(Aw,w)|,\ |p(Aw,Aw)|,\ |p(w,w)|,\ \frac{|p(Aw,Aw)|\cdot|p(w,w)|}{1+|p(Aw,Aw)|+|p(w,w)|},\ 2|p(Aw,w)|) \leq 0$$

We have $|p(Aw,Aw)| \leq |p(Aw,w)|$ and $|p(w,w)| \leq |p(Aw,w)|$, and the $\frac{|p(Aw,Aw)|\cdot|p(w,w)|}{1+|p(Aw,Aw)|+|p(w,w)|} \leq |p(Aw,w)|$.

Using the non-increasing property of Φ in arguments 2–5, we obtain:

$$\Phi(|p(Aw,w)|,\ |p(Aw,w)|,\ |p(Aw,w)|,\ |p(Aw,w)|,\ |p(Aw,w)|,\ 2|p(Aw,w)|) \leq 0$$

Let $\gamma = |p(Aw,w)|$. Then:

$$\Phi(\gamma,\gamma,\gamma,\gamma,\gamma,2\gamma) \leq 0$$

Applying condition (i) with $u = \gamma$ and $v = \gamma$:

$$\gamma \geq q\gamma \text{ with } q > 1$$

This forces $\gamma = 0$. Hence:

$$|p(Aw,w)| = 0 \implies p(Aw,w) = 0$$

From the partial metric definition, $p(Aw,Aw) \leq p(Aw,w) = 0$ implies $p(Aw,Aw) = 0$. Also $p(w,w) \leq p(Aw,w) = 0$ implies $p(w,w) = 0$.

Therefore:

$$p(Aw,Aw) = p(Aw,w) = p(w,w) = 0$$

By the partial metric definition, we get:

$$Aw = w$$

Then from (3.8), $Sw = Aw = w$. Thus:

$$Aw = Sw = w \qquad (3.9)$$

Similarly, substituting $x = x_0$ and $y = w$ into (3.1) and using the owc property of $\{B,T\}$, yields:

$$Bw = Tw = w \qquad (3.10)$$

Combining (3.9) and (3.10), we obtain:

$$Aw = Bw = Sw = Tw = w$$

Therefore, w is a common fixed point of A, B, S, T.

Finally, we have to prove that common fixed point is unique.

Assume, to the contrary, that there exists another point $z \in X$ with $z \neq w$ such that:

$$Az = Bz = Sz = Tz = z$$

Substitute $x = w$ and $y = z$ into inequality (3.1). Using $Sw = Aw = w$ and $Tz = Bz = z$,

Thus, inequality (3.1) becomes:

$$\Phi(|p(w,z)|,\ |p(w,z)|,\ |p(w,w)|,\ |p(z,z)|,\ \frac{|p(w,w)|\cdot|p(z,z)|}{1+|p(w,w)|+|p(z,z)|},\ 2|p(w,z)|) \leq 0$$

From the partial metric definition, we have:

$$|p(w,w)| \leq |p(w,z)| \text{ and } |p(z,z)| \leq |p(w,z)|$$

Also, the fifth term satisfies:

$$\frac{|p(w,w)|\cdot|p(z,z)|}{1+|p(w,w)|+|p(z,z)|} \leq \min(|p(w,w)|,|p(z,z)|) \leq |p(w,z)|$$

Since Φ is non-increasing in arguments 2 to 5, we can replace these arguments with the larger value $|p(w,z)|$ to obtain:

$$\Phi(|p(w,z)|,\ |p(w,z)|,\ |p(w,z)|,\ |p(w,z)|,\ |p(w,z)|,\ 2|p(w,z)|) \leq 0$$

Let $\delta = |p(w,z)|$. Then:

$$\Phi(\delta,\delta,\delta,\delta,\delta,2\delta) \leq 0$$

Applying condition (i) with $u = \delta$ and $v = \delta$:

$$\delta \geq q\delta \text{ with } q > 1$$

This implies $\delta(1-q) \geq 0$. Since $1-q < 0$, we must have $\delta \leq 0$. But $\delta = |p(w,z)| \geq 0$. Hence:

$$\delta = 0 \implies |p(w,z)| = 0 \implies p(w,z) = 0$$

From the partial metric definition, $p(w,w) \preceq p(w,z) = 0$ gives $p(w,w) = 0$, and $p(z,z) \preceq p(w,z) = 0$ gives $p(z,z) = 0$. Therefore:

$$p(w,w) = p(w,z) = p(z,z) = 0$$

By the partial metric definition, this forces:

$$w = z$$

This contradicts our assumption that $z \neq w$. Therefore, the common fixed point is unique.

Example

Example 1: Let $X = \{0,1,2\}$. Define $p: X \times X \to \mathbb{C}$ by $p(x,y) = |x-y| + i|x-y|$.

Let mappings:

$$A(x) = 0\ \forall x, B(x) = 0\ \forall x$$

$$S(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } 1 \\ 1 & \text{if } x = 2 \end{cases}$$

$$T(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } 2 \\ 1 & \text{if } x = 1 \end{cases}$$

Solution:

Let us verify (X,p) is a complete complex valued partial metric space.

For any $x,y,z \in X$:

- $p(x,x) = 0 \preceq p(x,y)$ since $|x-y| \geq 0$
- $p(x,y) = |x-y| + i|x-y| = p(y,x)$
- $x = y \iff p(x,x) = p(x,y) = p(y,y) = 0$
- $p(x,z) = |x-z| + i|x-z| \preceq (|x-y| + |y-z|) + i(|x-y| + |y-z|) = p(x,y) + p(y,z) - p(y,y)$ because $p(y,y) = 0$

Since X is finite, the space is complete.

To verify owc condition.

For $\{A,S\}$: Take $x = 0$. Then $A(S(0)) = A(0) = 0$ and $S(A(0)) = S(0) = 0$. Also $A(0) = S(0) = 0$.

Hence owc holds.
For $\{B, T\}$: Take $x = 0$. Then $B(T(0)) = B(0) = 0$ and $T(B(0)) = T(0) = 0$. Also $B(0) = T(0) = 0$.
Hence owc holds.

Choose $\Phi \in \Phi_6$.
Define $\Phi(u, v, a, b, c, d) = u - 2 \cdot \max\{v, a, b, c, d\}$ with $q = 2 > 1$.
Conditions

(i) $\Phi(u, v, v, v, v, w) \le 0 \Rightarrow |u| \ge \beta|v|\ for\ \beta > 1$
(ii) $\Phi(u, u, 1,1,1, u) \le 0 \Rightarrow u \ge 1$
are satisfied.

Verify inequality (3.1) for all $x, y \in X$.
For any $x, y \in X$, since $A(x) = B(y) = 0$, we have $p(Ax, By) = p(0,0) = 0$.Thus, the second argument of Φ is 0.
The inequality becomes:

$$|p(Sx, Ty)| - 2 \cdot \max\{0, |p(Sx, Ax)|, |p(By, Ty)|, \frac{|p(Ax, Sx)|.|p(By, Ty)|}{1 + |p(Ax, Sx)| + |p(By, Ty)|}, |p(Ax, Ty)| + |p(By, Sx)|\} \le 0$$

After examining all nine possible pairs $(x, y) \in X \times X$, we observe the following:
Case 1: When $|p(Sx, Ty)| = 0$
In this situation, the first argument of Φ is zero. The remaining arguments (2 through 6) are all non-negative real numbers. Therefore:

$$\Phi = 0 - 2 \times \max\{\text{non-negative numbers}\} \le 0$$

Thus, the inequality holds.
Case 2: When $|p(Sx, Ty)| = \sqrt{2}$
This occurs for the pairs (0,1), (1,1), (2,0), and (2,2). In each of these cases, the sixth argument $|p(Ax, Ty)| + |p(By, Sx)|$ equals $\sqrt{2}$ (or greater). Consequently:

$$\max\{v, a, b, c, d\} \ge \sqrt{2}$$

Hence:

$$\Phi = \sqrt{2} - 2 \times \max\{0, |p(Sx, Ax)|, |p(By, Ty)|, \frac{|p(Ax, Sx)|.|p(By, Ty)|}{1 + |p(Ax, Sx)| + |p(By, Ty)|}, |p(Ax, Ty)| + |p(By, Sx)|\} \le \sqrt{2} - 2\sqrt{2} = -\sqrt{2} \le 0$$

Thus, inequality holds for all $x, y \in X$.
$w = 0$ satisfies: $A(0) = 0$, $B(0) = 0$, $S(0) = 0$, $T(0) = 0$.
If $z \ne 0$ is another common fixed point, then $A(z) = 0 \ne z$ for $z = 1$ or 2. Hence no other fixed point exists. Therefore $w = 0$ is the unique common fixed point.

Example 2 Let $X = [0,1]$. Define $p: X \times X \to \mathbb{C}$ by:

$$p(x, y) = |x - y| + i|x - y|$$

Define mappings $A, B, S, T: X \to X$ as:

$$A(x) = \frac{x}{2}, B(x) = \frac{x}{2}, S(x) = x, T(x) = x \forall x \in X$$

Solution:

Verify (X, p) is a complete complex valued partial metric space.

For any $x, y, z \in [0,1]$:

- $p(x, x) = 0 \preceq p(x, y)$ since $\mid x - y \mid \geq 0$
- $p(x, y) = \mid x - y \mid + i \mid x - y \mid = p(y, x)$
- $x = y \iff p(x, x) = p(x, y) = p(y, y) = 0$
- $p(x, z) = \mid x - z \mid + i \mid x - z \mid \preceq (\mid x - y \mid + \mid y - z \mid) + i(\mid x - y \mid + \mid y - z \mid) = p(x, y) + p(y, z) - p(y, y)$

Thus (X, p) is a complex valued partial metric space.

To prove completeness

Let $\{x_n\}$ be a Cauchy sequence in (X, p). Then $\lim_{n,m\to\infty} p(x_n, x_m)$ exists. Since $p(x_n, x_m) = \mid x_n - x_m \mid + i \mid x_n - x_m \mid$, we have $\lim_{n,m\to\infty} \mid x_n - x_m \mid = 0$.

Thus $\{x_n\}$ is Cauchy in $([0,1], |\cdot|)$. As $[0,1]$ is complete, there exists $x \in [0,1]$ such that $\lim_{n\to\infty} \mid x_n - x \mid = 0$.

Then:

$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} (\mid x_n - x \mid + i \mid x_n - x \mid) = 0 = p(x, x)$$

Hence (X, p) is complete.

To verify owc condition.

For $\{A, S\}$: Take $\mathrm{x} = 0$. Then $\mathrm{A(S(0)) = A(0) = 0}$ and $\mathrm{S(A(0)) = S(0) = 0}$. Also $\mathrm{A(0) = S(0) = 0}$. Hence owc holds.

For $\{B, T\}$: Take $\mathrm{x} = 0$. Then $\mathrm{B(T(0)) = B(0) = 0}$ and $\mathrm{T(B(0)) = T(0) = 0}$. Also $\mathrm{B(0) = T(0) = 0}$. Hence owc holds.

Choose $\Phi \in \Phi_6$.

Define $\Phi(\mathrm{u, v, a, b, c, d}) = \mathrm{u} - 2 \cdot \max\{\mathrm{v, a, b, c, d}\}$ with $\mathrm{q} = 2 > 1$.

Verify inequality (3.1).

For any $\mathrm{x, y} \in [0,1]$:

- $\mathrm{p(Sx, Ty) = p(x, y) = \mid x - y \mid + i \mid x - y \mid \Rightarrow \mid p(Sx, Ty) \mid = \mid x - y \mid}$
- $\mathrm{p(Ax, By) = p(\frac{x}{2}, \frac{y}{2}) = \frac{|x-y|}{2} + i\frac{|x-y|}{2} \Rightarrow \mid p(Ax, By) \mid = \frac{|x-y|}{2}}$
- $\mathrm{p(Sx, Ax) = p(x, \frac{x}{2}) = \frac{|x|}{2} + i\frac{|x|}{2} \Rightarrow \mid p(Sx, Ax) \mid = \frac{|x|}{2}}$
- $\mathrm{p(By, Ty) = p(\frac{y}{2}, y) = \frac{|y|}{2} + i\frac{|y|}{2} \Rightarrow \mid p(By, Ty) \mid = \frac{|y|}{2}}$
- Fifth term = $\frac{|p(Ax,Sx)|\cdot|p(By,Ty)|}{1+|p(Ax,Sx)|+|p(By,Ty)|} = \frac{\frac{|x|}{2}\cdot\frac{|y|}{2}}{1+\frac{|x|}{2}+\frac{|y|}{2}} \leq \frac{|x||y|}{4}$
- Sixth term = $\mathrm{\mid p(Ax, Ty) \mid + \mid p(By, Sx) \mid = \mid p(\frac{x}{2}, y) \mid + \mid p(\frac{y}{2}, x) \mid}$

Note that

$$max\,\{\frac{\mid x - y \mid}{2}, \frac{\mid x \mid}{2}, \frac{\mid y \mid}{2}, \frac{\mid x \mid\mid y \mid}{4}, \mid y - \frac{x}{2} \mid + \mid x - \frac{y}{2} \mid \geq \mid x - y \mid$$

This is because the sixth term satisfies:

$$\mid y - \frac{x}{2} \mid + \mid x - \frac{y}{2} \mid \geq \mid x - y \mid$$

by the triangle inequality. Hence:

$$2 \times \max\{\frac{\mid x - y \mid}{2}, \frac{\mid x \mid}{2}, \frac{\mid y \mid}{2}, \frac{\mid x \mid\mid y \mid}{4}, \mid y - \frac{x}{2} \mid + \mid x - \frac{y}{2} \mid\ \} \geq \mid x - y \mid$$

Therefore:

$$\Phi = | p(Sx,Ty) | - 2 \times \max\{\frac{| x-y |}{2}, \frac{| x |}{2}, \frac{| y |}{2}, \frac{| x || y |}{4}, | y - \frac{x}{2} | + | x - \frac{y}{2} |\} = | x - y | - 2 \times \max\{\frac{| x-y |}{2}, \frac{| x |}{2}, \frac{| y |}{2}, \frac{| x || y |}{4}, | y - \frac{x}{2} | + | x - \frac{y}{2} | \} \leq 0$$

Thus inequality (3.1) holds for all $x, y \in X$.

We need $\mathrm{w} \in \mathrm{X}$ such that:

$$\mathrm{A(w) = w} \implies \frac{\mathrm{w}}{2} = \mathrm{w} \implies \mathrm{w} = 0$$
$$\mathrm{B(w) = w} \implies \frac{\mathrm{w}}{2} = \mathrm{w} \implies \mathrm{w} = 0$$
$$\mathrm{S(w) = w} \implies \mathrm{w = w} \text{ (always true)}$$
$$\mathrm{T(w) = w} \implies \mathrm{w = w} \text{ (always true)}$$

Thus $\mathrm{w} = 0$ is the common fixed point.

Suppose $\mathrm{z} \neq 0$ is another common fixed point. Then $\mathrm{A(z)} = \frac{\mathrm{z}}{2} = \mathrm{z}$ implies $\mathrm{z} = 0$, contradiction. Hence $\mathrm{w} = 0$ is unique.

Conclusion

This paper uses rational expansive mappings and newly defined implicit relations to establish a common fixed-point theorem in complex valued partial metric spaces. The existence and uniqueness of a common fixed point for four self-mappings are demonstrated by using the notion of occasionally weakly compatible mappings. By adding more adaptable and unified conditions, the results expand and generalize a number of fixed-point theory theorems. This work advances nonlinear analysis and offers a more comprehensive framework for future investigations in generalized metric spaces.

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