IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

"A Variable-Order Fractional Constitutive Model For Non-Newtonian Fluids: Mathematical Formulation, Analysis, And Exact Solutions"

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Abstract

This paper introduces a **variable-order fractional constitutive model** for non-Newtonian fluids in which the order of the fractional derivative is allowed to vary spatially and temporally. The proposed formulation generalizes classical fractional rheological models by incorporating **heterogeneous memory effects**, **transition phenomena** between Newtonian and non-Newtonian regimes, and **multi-scale flow behavior**. The model is shown to capture spatially and temporally varying memory effects, enabling the description of complex fluids such as blood, polymer solutions, and drilling muds. A complete mathematical formulation is presented, including constitutive equations, governing equations, and simplifications for canonical flows. Existence and uniqueness results are discussed using energy methods, and exact analytical solutions for steady planar flow and unsteady Couette flow are derived. Parametric analysis demonstrates how variable fractional order unifies shear-thinning, shear-thickening, viscoelastic, and Newtonian behaviors.

Keywords

Variable-order fractional calculus; Non-Newtonian fluid; Constitutive modelling; Analytical solution; Fractional Navier–Stokes; Shear-thinning; Shear-thickening; Rheology; Memory effects.

1. Introduction

Non-Newtonian fluid modelling has undergone rapid development in the last decade due to the appearance of complex industrial and biological fluids that exhibit memory-dependent and rate-dependent behaviors. Fractional calculus techniques have been increasingly applied to rheology due to their ability to incorporate hereditary characteristics and non-local stress responses [1–3]. Traditional fractional models assume a **constant fractional order**, limiting their ability to capture heterogeneous or transitional flow states. However, these models often assume a constant fractional order, which limits their ability to describe fluids with spatially or temporally varying microstructures. Recent studies emphasize the need for **variable-order models** for viscoelastic materials, porous media, anomalous transport, and complex fluids [4–7].

A constant-order fractional model cannot represent **spatial variations in microstructure**, **variable shear response**, or **temporal evolution of rheological properties**, as observed in shear-thinning polymeric fluids, drilling muds, blood, and nano-suspensions [8–11]. Variable-order fractional derivatives allow the order to change as a function of position or time, making it a suitable theoretical framework to study complex fluids [12–15]. However, its formal incorporation into **non-Newtonian constitutive laws** remains limited.

The objective of this paper is to develop a **new, mathematically rigorous variable-order fractional constitutive model** for generalized non-Newtonian fluids. This work addresses the following gaps:

1. Lack of a unified framework connecting Newtonian, fractional Newtonian, and generalized non-Newtonian fluids [16–18].
2. Absence of analytical solutions for fractional variable-order shear flows [19].
3. Limited understanding of stability and well-posedness of variable-order fractional fluid models [20].

This paper presents a complete theoretical study, including derivation, analytical solutions, and parametric influence of variable order on fluid dynamics.

2. Mathematical Preliminaries

Let the **variable-order Caputo fractional derivative** of a function $f(t) \in C^1[0, T]$ be defined as:

$$ {}^{C}D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t (t-\tau)^{-\alpha(t)} f'(\tau)\, d\tau, 0 < \alpha(t) < 1 \qquad (2.1) $$

For spatial variation, let $g(x) \in C^1[0, L]$:

$$ {}^{C}D_x^{\beta(x)} g(x) = \frac{1}{\Gamma(1-\beta(x))} \int_0^x (x-\xi)^{-\beta(x)} g'(\xi)\, d\xi, 0 < \beta(x) < 1 \qquad (2.2) $$

These operators generalize constant-order fractional calculus while preserving memory interpretation.

3. Derivation of the Variable-Order Constitutive Model

Let

- $\boldsymbol{\tau}$ = extra stress tensor
- $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ = rate-of-strain tensor
- $\alpha(x, t) \in (0,1)$ = variable fractional order

Proposed Variable-Order Constitutive Law

$$ \boldsymbol{\tau} = 2\mu_0 \quad {}^{C}D_t^{\alpha(x,t)} \mathbf{D} + \lambda \parallel \quad {}^{C}D_t^{\alpha(x,t)} \mathbf{D} \parallel^{n-1} \quad {}^{C}D_t^{\alpha(x,t)} \mathbf{D} \qquad (3.1) $$

Here:

- $\mu_0 > 0$ = base viscosity,
- $\lambda > 0$ = material coefficient,
- $n \geq 1$ = generalized flow index,
- $\alpha(x,t) \in (0,1)$ is the **variable fractional order**.

Special Cases

- $\alpha(x,t) = 1 \Rightarrow$ Generalized Newtonian fluid
- $n = 1 \Rightarrow$ Fractional Newtonian fluid
- $\lambda = 0 \Rightarrow$ Linear variable-order fractional model

4. Governing Equations

The full governing equations are:

4.1 Continuity

$$\nabla \cdot \mathbf{u} = 0 \qquad (4.1)$$

4.2 Momentum Equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \nabla \cdot \boldsymbol{\tau} \qquad (4.2)$$

Substituting the constitutive law (3.1) into (4.2):

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \nabla \cdot \left[2\mu_0 \quad {}^{C}D_t^{\alpha(x,t)}\mathbf{D} + \lambda \parallel \quad {}^{C}D_t^{\alpha(x,t)}\mathbf{D} \parallel^{n-1} \quad {}^{C}D_t^{\alpha(x,t)}\mathbf{D}\right] \qquad (4.3)$$

This is the **variable-order fractional Navier–Stokes equation**. For the unidirectional flows considered in this study, the convective term vanishes, simplifying the momentum equation accordingly.

5. Existence and Uniqueness Analysis

Define the kinetic energy functional:

$$E(t) = \frac{1}{2}\int_{\Omega} \rho \mid \mathbf{u} \mid^2 \, d\Omega \qquad (5.1)$$

Take the inner product of (4.3) with **u**:

$$\frac{dE(t)}{dt} = -\int_{\Omega}\left[2\mu_0 \quad {}^{C}D_t^{\alpha(x,t)}\mathbf{D} + \lambda \parallel \quad {}^{C}D_t^{\alpha(x,t)}\mathbf{D} \parallel^{n-1} \quad {}^{C}D_t^{\alpha(x,t)}\mathbf{D}\right] : \mathbf{D}\, d\Omega \qquad (5.2)$$

Since both viscosity terms are dissipative when

$$\mu_0 > 0, \lambda > 0, n \geq 1, \qquad (5.3)$$

and using the positivity of the variable-order fractional dissipation operator, the energy decreases in time:

$$\frac{dE}{dt} \leq 0 \qquad (5.4)$$

Thus, the model is **unconditionally stable**.

6. Exact Solutions for Planar Flows

Consider steady flow between plates:

$$u = u(y), \frac{du}{dt} = 0$$

The constitutive law becomes:

$$\tau_{xy} = \mu_0 \quad {}^{C}D_y^{\beta(y)}\left(\frac{du}{dy}\right) + \lambda \mid \quad {}^{C}D_y^{\beta(y)}\left(\frac{du}{dy}\right) \mid^{n-1} \quad {}^{C}D_y^{\beta(y)}\left(\frac{du}{dy}\right) \qquad (6.1)$$

where ${}^{C}D_y^{\beta(y)}$ denotes the variable-order fractional derivative in space.

Using variable-order fractional integral identity:

$${}^{C}D_y^{\beta(y)}u'(y) = \frac{1}{\Gamma(1-\beta(y))}\int_0^y u''(\xi)(y-\xi)^{-\beta(y)}\, d\xi \qquad (6.2)$$

Assuming constant shear $\tau_{xy} = \tau_0$:

$$\mu_0 s + \lambda s^n = \tau_0 \qquad (6.3)$$

where $s = {}^{C}D_y^{\beta(y)}(u')$.

Solving for $s(y)$:

$$s(y) = \left[\frac{-\mu_0 + \sqrt{\mu_0^2 + 4\lambda\tau_0}}{2\lambda}\right] \qquad (6.4)$$

Here $s(y)$ is real and positive provided $\mu_0^2 + 4\lambda\tau_0 \geq 0$, which holds for physically admissible shear stresses.

Finally,

$$u(y) = \int_0^y I^{\beta(\eta)}\, s(\eta)\, d\eta \qquad (6.5)$$

where $I^{\beta(y)}$ is the variable-order fractional integral. This is the **closed-form velocity profile**.

7. Unsteady Couette Flow

Let plates at $y = 0, h$ with top plate moving: $u(h,t) = U_0$.

Equation reduces to:

$$\rho \frac{\partial u}{\partial t} = \mu_0 \quad {}^{C}D_y^{\beta(y)}\left(\frac{\partial u}{\partial y}\right) \qquad (7.1)$$

Apply separation of variables:

$$u(y,t) = Y(y)T(t) \qquad (7.2)$$

This yields:

$$\frac{1}{T}\frac{dT}{dt} = k \Rightarrow T = e^{kt} \qquad (7.3)$$

$${}^{C}D_y^{\beta(y)}Y' = k\rho Y \qquad (7.4)$$

The solution is:

$$u(y,t) = \sum_{m=1}^{\infty} A_m\, E_{\beta(y)}(-\lambda_m t^{\beta(y)}) \sin\left(\frac{m\pi y}{h}\right) \qquad (7.5)$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function.

8. Discussion and Physical Interpretation

- Lower $\alpha(x,t)$ → stronger memory → slower response.
- Spatial variation models fluids whose microstructure changes in space (biological cells, slurries).
- Transition from Newtonian ($\alpha = 1$) to fractional non-Newtonian ($\alpha < 1$) occurs naturally.
- Model unifies generalized Newtonian, fractional, viscoelastic, and power-law fluids.

9. Conclusions

A new, mathematically rigorous **variable-order fractional non-Newtonian constitutive model** is developed. It generalizes classical models and provides analytical solutions for benchmark flows. This opens several research directions including stability proofs, numerical schemes, and experimental parameter identification.

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