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"Summable Families in nuclear Locally Convex Space"

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ABSTRACT

In this paper, we have to give systematic account of the most important ideas, methods and results of Summable Families in nuclear Locally Convex Space. We also found that the combining the topology with linear structure and algebraic structure under the normal compatibility conditions. In this process, we have also study the metrizable vector spaces in which topology can be defined by metric by considering countable bases of neighborhoods. We will proceed to study the Summable Families in nuclear Locally Convex Space .The Summable Families in details and we will try to establish some new and useful results.

KEYWORDS: Functional analysis, Topological vector spaces, nuclear locally convex spaces, Tensor products.

Introduction

Below is a reasonably self-contained explanation of **summable families in nuclear locally convex spaces**, including what the notion means, why it is natural, and how it interacts with nuclearity.

Locally convex space

A locally convex space (LCS) is a topological vector space whose topology comes from a family of seminorms. All standard spaces used in functional analysis (Banach, Fréchet, Montel, Schwartz spaces, etc.) are locally convex.

Nuclear space

A locally convex space EEE is nuclear if every continuous linear map from EEE into an arbitrary Banach space is nuclear in the sense of Grothendieck (the operator behaves like a trace-class operator). Equivalently, tensor products behave "as if finite dimensional," and this leads to very strong summability and compactness properties.

Summable Families

Let E be an arbitray locally convex space, In the sequel we shall write

$$l_I^1[E] = l_I^1\{E\} \text{ or } l_I^1(E) = l_I^1\{E\}$$

If the two linear space $l_I^1[E]$ or $l_I^1(E)$ coincide with $l_I^1\{E\}$

without reference to the topology. If the ε-topology coincides with the π-topology as well we shall use the notation

$$l_I^1[E] \equiv l_I^1(E) \equiv l_I^1\{E\}.$$

Preposition: If E is a nuclear locally convex space, then for each index set I we have identity

$$l_I^1[E] = l_I^1\{E\} \text{ or } l_I^1(E) = l_I^1\{E\}.$$

Proof: Because We Know for each zero neighborhood U∈ $\mu(E)$ a zero neighborhood V∈ $\mu(E)$ and a sequence of linear forms $a_n \in E'$ with

$$\sum_N p_{V^\circ}(a_n) \equiv 1$$

For which we have the inequality

$$p_U(x) \equiv \sum_N |\langle x, a_n \rangle| \quad \text{for all } x \in E$$

Since

$$\sum_I |\langle x, a_n \rangle| \equiv p_{V^\circ}(a_n)\, \varepsilon_V [x_i, I]$$

Is true for all families $[x_i, I] \in l_I^1[E]$, we have the inequality

$$\pi_U[x_i, I] \equiv \sum_I p_U(x_i) \equiv \sum_N \sum_I |\langle x_i, a_n \rangle| \equiv \varepsilon_V [x_i, I]$$

Consequently, all weakly summable families in E are also absolutely summable and we have

$$l_I^1[E] \equiv l_I^1(E) \equiv l_I^1\{E\}.$$

We have at the same time proved that the ε-topology is finer than the π-topology so that the two topologies must coincide.

As a converse to the assertion of above Proposition. We have the

Proposition: Each locally convex space E for which the identity

$$l_i^1[E] \equiv {}_i^1 l\,\{E\} \quad (1) \qquad l_i^1(E) \equiv \{E\} \quad (1)$$

Holds for some infinite index set I, is nuclear.

Proof: in the first place, since the inclusion

$$l_i^1[E] \subset l_i^1(E) \subset {}_i^1 l\,\{E\}$$

Hold for every locally convex space, (2) always follows form (1).

If (2) is valid then the ε-topology coincides with the π-topology on $l_I^1(E) \equiv {}_I^1 l\,\{E\}$ so that for each zero neighborhood U$\varepsilon\,\mu(E)$ there is a zero neighborhood V$\varepsilon\,\mu(E)$ for which we have

$$\pi_U[x_i, I] \equiv {}_V\varepsilon\,[x_i, I] \quad \text{for} \quad [x_{i,}I]\ \varepsilon\ {}_i^1 l\,(E).$$

We now consider an arbitrary finite family $[x_n(V),\mu]$ from E(V). If I ={i_1 ………, i_k } is a finite subset of I which has exactly as many elements as the setμ = {n_1 ………, n_k }, we form the families $[y_{i,}I]$ belonging to$l_I^1(E)$ with y_i = 0 for I not belongs to I and y_{ih}=x_{nk} for h=1, …….k. than because of the identities

$$\sum_n p[x_n(U)] \equiv \sum_n p_U(x_n) \equiv \sum_I p_U(y_I) \equiv \pi_U[y_i, I]$$

and

$$\sup\left\{\sum_n |\langle x_n(V), b\rangle| : b \in V^1\right\} \equiv \sup_I \left\{\sum_I |\langle y_i, b\rangle| : b \in V^1\right\} \equiv \varepsilon_V[y_i, I]$$

We have the inequality

$$\sum_n p[x_n(U)] \equiv \sup_n \left\{\sum_n |\langle x_n(V), b\rangle| : b \in V^1\right\}$$

Hence the canonical mapping from E(V) onto E(U) is absolutely summing, and we have shown than the fundamental system U(E) has property (N').

By combining the result of above two section

we obtain the

Theorem. A locally convex space E is nuclear if and only if for some resp. each, infinite index set I the identity

$$l_I^1[E] \equiv l_I^1\{E\} \quad or \quad l_I^1(E) \equiv l_I^1\{E\}$$

holds $l_I^1[E] \equiv l_I^1\{E\} \quad or \quad l_I^1(E) \equiv \{E\}$

For a matric locally convex space E the identity

$$l_i^1[E] \equiv l_I^1\{E\} \quad or \quad l_I^1(E) \equiv l\{E\}$$

Implies that the ε -topology is finer than the π-topology

Consequently the two topologies coincide and we have

$$l_i^1[E] \equiv l_I^1\{E\} \quad or \quad l_I^1(E) \equiv l\{E\}$$

Therefore, we have metric locally convex space the following sharpening of above theorem which also holds for dual metric locally convex space because of above theorem

Theorem: A metric or dual metric locally convex space E is nuclear if and only for some, resp. each, infinite index set I the identity

$$l_i^1[E] \equiv l_I^1\{E\} \quad or \quad l_I^1(E) \equiv l\{E\}$$

holds

The question of the generalisation of the previous theorem is posed as a

Problem. What properties must a locally convex space E have so that from the validity of the identity

For an infinite index set I it follows that E is nuclear?

We now extent the previous result of this section to dual nuclear space. For this we use the

Lemma. It E is a dual nclear locally convex space than for each bounded subset **B** of $l_i^1[E]$ there is a bounded subset B ϵ 𝔅(E) with

$$\sum_{I} p_B(x_I) \equiv 1 \quad \text{for all } [x, I]$$

Proof : By hypothesis the number

$$\sigma_U = \sup\{y\ [x_i, I]: [x_I, I] \in B$$

Is bounded for each zero neighborhood U ε (E) and we can form the set

$$A = \{\sigma_U \quad \varepsilon\ (E)\}$$

Which belongs to consequently, we have

$$\sum_{I} {}_I x_I \ \epsilon \in \quad \text{for } [x_I, I] \in B \quad |\ {}_I| \equiv 1 \text{ a } \iota \epsilon\ (I)$$

And the inequality

$$\epsilon\ [x_I, I] \equiv 1 \quad \text{for all } [x_I, I]$$

holds.

We know determine a set Bε 𝔅 (E) with A B such that the canonical mapping E(A,B) is absolutely summing. Since we can always make the π-norm of E(A,B) smaller than 1, the inequality

$$\pi\ [x_{i,} I] \equiv \epsilon\ {}_I[x, I] \quad \text{for all } [x_{i,} I_I][\varepsilon\ l\ E\ (\ )]$$

is vaild and the estimate

$$\sum_{I} p_B(x_I) \equiv 1 \quad \text{for all } [x, I]$$

Is proved.

Proposition. If E is a dual nuclear locally convex space, then for each index set I the identity

$$\iota_I^1[E] \equiv \iota_I^1(E) \equiv \iota_I\{E\} = \iota_I\langle E\rangle$$

holds

Proof: Since each weakly summable family [x, I] from E can be regarded as a bounded singleton from ι_I [E] there is by the lemma just proved a set B ϵ 𝔅 (E) with

$$\sum_{I} p_B(x_I) \equiv 1$$

Consequently, all weakly Summable families from E are also totally Summable and we have

$$l_I^1[E] \equiv l_I^1(E) \equiv l_I^1\{E\} = l_I\langle E\rangle$$

Proposition. Each dual nuclear space E has property(B).

Proof. Since each bounded subset B of can also be considered as a bounded subset of there exists by above Lemma a bounded subset B ϵ $\mathfrak{B}$ (E) with

$$\sum_{I} p_B(x_I) \equiv 1 \quad \text{for all } [x_I, I]$$

Consequently the locally convex space has property (B).

As a converse of the above statements, we have the

Proposition: Each locally convex space E with property (B) for which the identity

$$l_i^1[E] \equiv l_i^1\{E\} \quad (1) \quad (\; l_i^1(E) \equiv \{E\} \quad (1)$$

holds for some infinite index set I is dual nuclear.

Proof: To begin with, we recall that since the inclusion

$$l_i^1\{E\} \subset l_i^1(E) \subset l_i^1[E]$$

is true for every locally convex space E, (2) always from (1).

For an arbitrary set A ϵ $\mathfrak{B}$ (E)

$$B = \{[x_n, N] \in l_N^1[E(\;)] : \varepsilon_n[x_n, N] \equiv\}$$

is a bounded subset of $l_N^1[E]$. Since from (2) it follow that the identity mapping from E into itself is absolutely asumming , B must by the above theorem be a bounded subset of $l_N^1\{E\}$ as well. Therefore, there exists a set Bϵ $\mathfrak{B}$ (E) with

$$\sum_{I} p_B(x_i) \equiv 1 \quad \text{for all } [x_i]$$

Form this statement it follows that te inequality

$$\pi_B[x_n,)] \equiv \epsilon_n[x_n,] \quad \text{for all } [x_n] \in l_I^1 \; E(\epsilon$$

is valid, Consequently, the canonical mapping from E(A) into E(B) is absolutely summing and we have shown that the fundamental system $\mathfrak{B}$ (E) has property (N).

By combining the result, we obtain the

Theorem. A locally convex space E is dual nuclear if and only if has property (B) and for some, resp. each, infinite index set I the identity

$$l_I^1[E] = l_I\{E\} \text{ or } l_I(E) = l_I\{E\}$$

Holds

As a special case we obtain by means of the following counterpart to theorem

Theorem: A matric or dual metric locally convex space E is dual nuclear if and only if for some, resp. each, infinite index set I the identity

$$l_I^1[E] = l_I\{E\} \text{ or } l_I(E) = l_I\{E\}$$

is valid.

Summary Statement

In a nuclear locally convex space, a family is summable if the sum of its seminorms is finite; this implies unconditional convergence. Nuclearity ensures that notions of summability that can differ in general LCS (absolute vs. unconditional vs. net convergence) become equivalent, making summation behave almost like in finite-dimensional vector spaces.

References

- **A. Grothendieck** – Produits tensoriels topologiques et espaces nucléaires (foundational monograph)
- **DE GRUYTER** –Nuclear Locally Convex Spaces
- **K. Floret** – Nuclear Spaces
- **Jarchow** – Locally Convex Spaces **H. H. Schaefer / M. Puhl** – Topological Vector Spaces
- **Trèves** – Topological Vector Spaces, Distributions and Kernels