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Structural Parallels Of Weakly Commutative Near Rings

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ABSTRACT

Near-rings constitute a significant generalization of rings and semigroups, offering a flexible algebraic framework for both theoretical exploration and applied contexts. A left near – ring N is called weak commutative if xyz = xzy for every x,y,z ϵ N. A left near – ring N is called pseudo commutative if xyz = zyx for every x,y,z ϵ N. A left near – ring N is called quasi weak commutative near- ring if xyz = yxz for every x,y,z ϵ N. We establish fundamental results concerning the existence, uniqueness, and characterization of ideals within these classes and provide necessary and sufficient conditions under which a left Boolean near-ring admits a quasi-weak commutative structure.This findings not only extend existing results in near-ring theory but also highlight potential avenues for applications in automata theory, cryptographic systems, and error-correcting codes. These results enrich the structural understanding of near-rings, provide new tools for analysing their ideal theory, and suggest applications in coding theory and cryptography where idempotent-based algebraic systems are of particular interest.

KEYWORDS: Quasi-weak commutative, Boolean near ring, Left near ring.

I PRELIMINARIES

Definition 1.1 [5]

A (right)Near ring is a set N together with two binary operations "+" and "." such that

a) (N,+) is a group (not necessarily abelian),
b) (N,.) is a semigroup
c) $(n_1 + n_2)\, n_3 = n_1n_3 + n_2n_3$, for all n_1,n_2,n_3 in N (right distributive law)

Definition 1.2 [1]

The function f: R → R' is said to be a homomorphism if:

(i) f(a+b) = f(a) + f(b)
(ii) f(ab) = f(a) f(b), for all a,b in R

Definition 1.3 [1]

The function f: R → R' is said to be an anti-homomorphism if:

(i) f(a+b) = f(b) + f(a)

(ii) f(ab) = f(b) f(a), for all a,b in R

Definition 1.4 [2]

An element a in G is said to be an idempotent element if $a^2 = a$ and the set of all idempotents is denoted by E.

Definition 1.5 [4]

A near ring R is said to be zero commutative if ab = 0 implies ba = 0, for all a,b in R.

Definition 1.6 [6]

A near ring N is said to be boolean if $a^2 = a$ for every a in N.

Proposition 1.7 [5]

Let N be a near ring.

a) N is abelian and N is commutative if and only if N is a commutative ring.
b) N is abelian and N is distributive if and only if N is a ring.
c) $N^2 = N$ and N is distributive implies that N is a ring.

Definition 1.8 [4]

Let N be a regular quasi weak commutative near – ring. Then every N sub group is an ideal N = Na = Na2 = aN = aNa for all a in N

Definition 1.9 [4]

A near ring N is said to be weak commutative if xyz = xzy for all x,y,z in N

Definition 1.10 [4]

Let N be a regular quasi weak commutative near – ring. Then

(i) A = $\sqrt{AA}$, for every N – subgroup A of N.

(ii) N is reduced.

(iii) N has insertion of factors property (IFP)

Definition 1.11 [2]

A near ring N is said to be commutative if ax = xa for all a,x in N.

Theorem 1.12 [5]

i)If I is an ideal of N, then the canonical mapping f: N $\rightarrow$ N\I (defined by f(n) = (n + I)) is a near ring epimorphism. Also, N\I is a homomorphic image of N.

ii) Conversely, if h: N $\rightarrow$ N^1 is an epimorphism, then ker(h) is an ideal of N, and N \ ker(h) is isomorphic to N^1.

Proposition 1.13 [2]

Let N be a near ring.

a) n ϵ N is right cancellable if and only if n is not a zero divisor;
b) If n ϵ N_0 is left cancellable then n is not a left zero divisor. But converse need not to be true.
c) If N ϵ η_0 then the left cancellation law implies the right one.

Definition 1.14 [5]

A near ring N is said to be left self – distributive if xyz = xyxz, for all x,y,z in N.

Definition 1.15 [5]

A near ring N is said to be right self – distributive if xyz = xzyz, for all x,y,z in N.

Definition 1.16 [4]

N is said to be subcommutative, if aN = Na for all a ϵ N

Definition 1.17 [2]

An element a in N is called central if ax = xa for all x in N.

Definition 1.18 [2]

A near ring N is said to be anti-boolean if a^2 = - a for every a in N.

Definition 1.19 [5]

A non – zero symmetric near – ring N has Intersection of factors Property (IFP) if and only if (O:S) is an ideal for any subset S of N.

Proposition 1.20 [3]

If N is a Boolean (left) near – ring, then for any a,b ϵ N, ab = 0 => ba = 0.a.

Proposition 1.21 [3]

If N is a Boolean (left) near – ring, then for any x,y ϵ N, xyx = yx.

Proposition 1.22 [3]

If N is anti – Boolean left near – ring then for any a,b ϵ N, ab =0⇒ ba = - 0a.

II MAIN RESULTS

Theorem 2.1

If N is a Boolean (left) near ring, then xyx = 0.x, for all x in N.

Proof:

Given that N is a Boolean (left) near ring, then for any x,y ϵ N,

xy = 0 implies that yx = 0.x (By Proposition 1.20)

To Prove: xyx = 0.x, for all x in N.

xyx = $(xyx)^2$ = (xyx) (xyx)

= $xy(x)^2yx$

= xyxyx

= xy(xy)x

-= xy(0)x (By Proposition 1.20)

= 0.x

xyx = 0.x, for all x,y in N

Theorem 2.2

If N is a Boolean (left) weak commutative near ring and x.y = 0, then y.x = 0 for all x, y in N.

Proof

Given that N is a Boolean left near ring, (i.e,) xy = 0

Then xyx = yx for all x,y in N (By Proposition 1.21)

Consider yx

yx = xyx (By Proposition 1.21)

= xxy

= x^2y

= xy

= 0

This implies, yx = 0, for all x,y in N.

Theorem 2.3

If N is a Boolean (left) near ring, then for any x,y in N, xyx = xy.

Proof

Let x,y ϵ N

Now, $xy(xyx – xy) = xyx^2y – (xy)^2$

$= xyxy – (xy)^2$

= 0

Then (xyx – xy) xy = 0.xy --------(1) (By Proposition 1.20)

⇒ (xyx – xy) xy = 0.xy

Also, $xyx (xyx\text{-}xy) = xyx^2yx – xyxyx$

= xyxyx – xyxyx

= 0

⇒ xyx (xyx-xy) = 0

Then (xyx – xy) xyx = 0.xyx --------(2) (By Proposition 1.20)

Now, $xyx – xy = (xyx – xy)^2$

= (xyx – xy) (xyx-xy)

= (xyx -xy) xyx – (xyx – yx) xy (By (1) and (2)

$= 0.\ xyx - 0.\ xy$

$\Rightarrow xyx - xy = 0\ (xyx - xy)$ -------(3)

Now, $0 = x\ (xyx - xy)$

$= x\ (0(xyx-xy))$ (By (3))

$= 0(xyx-xy)$

$= xyx - xy$ (By (3))

$\Rightarrow xyx = xy$, for all x,y in N.

Theorem 2.4

If N is anti-boolean (left) weak commutative near ring and $xy = 0$, then $yx = y0$, for all x,y in N.

Proof

Given that N is anti-boolean, then $x^2 = x$ for all x in N

N is weak commutative, $xyz = xzy$, for all x,y,z in N

Consider yx

$yx = (yx)^2$

$= (yx)\ (yx)$

$= y\ (xyx)$

$= y\ (xxy)$

$= yx^2 y$

$= -yxy$

$\Rightarrow\ yx = y(xy)$

$= y0$

$\Rightarrow yx = y0$, for all x,y in N.

Theorem 2.5

If N is anti-boolean (left) near ring, then for any x, y in N, $xyx = -xy$

Proof

Given that that N is anti-boolean, then $x^2 = x$ for all x in N

Let $x,y \in N$

$xy\,(xyx + xy) = xyx^2y + (xy)^2$

$= -\,xyxy + (xy)^2$

$= 0.$

Then $(xyx + xy) = -\,0xy$ ------(1) (By Proposition 1.22)

Also, $xyx\,(xyx + xy) = xyx^2yx + xyxyx$

$= -xyxyx + xyxyx$

$= 0$

Then $(xyx + xy)\,xyx = -\,0xyx$ -------(2) (By Proposition 1.22)

Now, $(xyx + xy) = (xyx + xy)^2$

$= (xyx + xy)\,(xyx + xy)$

$= (xyx + xy)\,xyx - (xyx + xy)\,xy$

$= -0xyx - 0xy$ (By (1) & (2))

$\Rightarrow xyx + xy = 0\,(xyx + xy)$ -------(3)

Now, $0 = x\,(xyx + xy)$

$= x.0\,(xyx + xy)$ (By (3))

$= 0.\,(xyx + xy)$

$= (xyx + xy)$ (By (3))

$\Rightarrow xyx = -\,xy$, for all x,y in N.

Theorem 2.6

If N is a Boolean (left) near ring, then for any x,y in N, $x^my^nx^m = x^my^n$ where $m \geq 1$, $n \geq 1$ and m,n are fixed integers.

Proof

Let $x,y \in N$

Consider $x^my^n (x^my^nx^m - x^my^n)$

$x^my^n (x^my^nx^m - x^my^n) = x^my^n x^{2m}y^n - x^my^nx^my^n$

$= x^my^nx^my^n - x^my^nx^my^n$

$= 0$

Then, $(x^my^nx^m - x^my^n) x^my^n = 0. x^my^n$ ------(1) (By Proposition 1.20)

Also, $x^my^nx^m (x^my^nx^m - x^my^n) = x^my^n x^{2m}y^nx^m - x^my^nx^my^nx^m$

$= x^my^nx^my^nx^m - x^my^nx^my^nx^m$

$= 0$

$(x^my^nx^m - x^my^n) x^my^nx^m = 0. x^my^nx^m$ -------(2) (By Proposition 1.20)

Now, $(x^my^nx^m - x^my^n) = (x^my^nx^m - x^my^n)^2$

$= (x^my^nx^m - x^my^n) (x^my^nx^m - x^my^n)$

$= (x^my^nx^m - x^my^n) x^my^nx^m - (x^my^nx^m - x^my^n) x^my^n$

$= 0. x^my^nx^m - 0. x^my^n$ (By (1) and (2))

$= 0 (x^my^nx^m - x^my^n)$ -----(3)

Now, $0 = x^m (x^my^nx^m - x^my^n)$

$= x^m 0 (x^my^nx^m - x^my^n)$ (By (3))

$= 0 (x^my^nx^m - x^my^n)$

$= (x^my^nx^m - x^my^n)$ (By (3))

$\Rightarrow x^my^nx^m = x^my^n$, where $m \geq 1$, $n \geq 1$ and for every x,y in N.

Theorem 2.7

If N is anti-boolean(left) near ring, then for any x,y in N, $x^my^nx^m = 2x^my^nx^my^n (x^m + 1)$, where $m \geq 1$, $n \geq 1$ and m,n are fixed inegers.

Proof

Let x,y ϵ N.

$x^my^n(x^my^nx^m – ynxm) = x^my^nx^{2m}y^n – x^my^nx^my^n$

$= - x^my^nx^my^n – x^my^nx^my^n$

$= - 2x^my^nx^my^n$ ------(1)

Now, $x^my^nx^m(x^my^nx^m – x^my^n) = x^my^nx^{2m}y^nx^m – x^my^nx^my^nx^m$

$= - x^my^nx^my^nx^m – x^my^nx^my^nx^m$

$= - 2\, x^my^nx^my^nx^m$ -----(2)

$x^my^nx^m - x^my^n = - (x^my^nx^my^nx^m - x^my^n)^2$

$= - [(x^my^nx^m - x^my^n) (x^my^nx^m - x^my^n)]$

$= - [\, x^my^nx^m(x^my^nx^m - x^my^n)- x^my^n (x^my^nx^m - x^my^n)]$

$= - [-2\, x^my^nx^my^nx^m - 2\, x^my^nx^my^n]$ (By (1) and (2))

$= 2\, x^my^nx^my^nx^m + 2\, x^my^nx^my^n$

$= 2\, x^my^nx^my^n (x^m+1)$

Hence the proof.

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