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Some Results On Hilbert-Schmidt Operators

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ABSTRACT

In this article, we shall prove a few of the most useful results on Hilbert-Schmidt Operators in our own technique. We shall also prove the following inclusions between different classes of operators on ,, each of which is a two-sided, self-adjoint ideal of operators in p(,).

$$finite\ rank \rightarrow trace\ class \rightarrow Hilbert - Schmidt \rightarrow compact \rightarrow bounded$$

Key words : Hilbert-Schmidt Operators, Trace Class operators, Compact operators, Bounded operators.

Introduction

In 1966 some situations , denotes an abstract Hilbert space. In others we assume that , $= L^3(X, dx)$, where X is a locally compact Hausdorff space and there is a countable basis to its topology. In this case we always assume that the Borel measure dx has support equal to X.

We warn that we have made a tiny selection from the many classes of operators that have been found useful in various contexts. In all of the proofs we shall assume that , is infinite dimensional & separable. The finite dimensional proofs are often simpler.

Definition 1. Let E and F be Hilbert spaces and let $[e_\alpha]$ and $[f_\beta]$ denote respectively orthonormal bases on E and F. A continuous linear map $A : E \rightarrow F$ is called a Hilbert-Schmidt operator if

$$K(A)^2 = \sum_{\alpha,\beta} {}^*\left(A_\alpha . f_\beta\right)^{*^2} < \infty$$

The number $K(A)$ is called the Hilbert-Schmidt norm.

Parseval's relation yields :

$$K(A)^2 = \sum_{\alpha,\beta} {}^*\left(Ae_\alpha . f_\beta\right)^{*^2}$$

$$= \sum_{\alpha} {}^{**}\left(Ae_\alpha\right)^{**^2}$$

$$= \sum_{\beta} {}^{**}\left(A' f_\beta\right)^{**^2}$$

Where A^* is the transpose of A.

Definition 2. A linear operators $K : X \to Y$ is called compact, if there exists a neighbourhood U of origin in X, whose image set $K(U)$ is relatively bounded subset of Y.

Definition 3. Let X and Y be topological vector spaces. A linear operators $K : X \to Y$ is called bounded if there exists a neighbourhood U of origin in X, whose image set $K(U)$ is relatively compact.

Definition 4. Important subclasses of compact operators are the trace class or nuclear operators.

Lemma 1. If $[e_n]_{n=1}^{\infty}$ and $[f_n]_{n=1}^{\infty}$ are two complete orthonormal sets in a Hilbert space , and A is a bounded operator on , then

$$\sum_{n=1}^{\infty} **(Ae_n)**^2 = \sum_{m,n=1}^{\infty} **(Ae_n, f_m)**^2 = \sum_{m=1}^{\infty} **(A^* f_m)**^2$$

Where the two sides converge or diverge together. It follows that the values of the two outer sums do not depend upon the choice of either orthonormal set.

Proof. Simplifiers the middle sum two different ways.

We say that A is Hilbert-Schmidt operator so that $A \in C_2$ if the above series converge, and write

$$**(Ae_n)**_2^2 := \sum_{N-1}^{\infty} **(Ae_n)**^2$$

The Hilbert-Schmidt norm $**\cdot**_2$ is also called the Frobenius norm. The notation $**\cdot**_2$ is also used.

Lemma 2. Kvery Hilbert-Schmidt operators A acting on a Hilbert space , is Compact.

Proof. Given two vectors $e, f \in$, we can make them the first terms of two complete orthonormal sequences. This implies

$$*(Ae, f)*^2 \le \sum_{m,n=1}^{\infty} *(Ae_n, f_m)*^2 = **(A)**_2^2$$

Since e, f are arbitrary we deduce that $**(A)** \le **(A)**_2$.

For any positive integer N one may write $A = A_N + B_N$ where

$$A_{N_g} = \sum_{m,n-1} (Ae_n, f_m)\langle g, e_N\rangle f_m.$$

For all $g \in$,. Since A_N is finite rank and

$$**B_N**^2 \le **(B_N)**_2^2 = \sum_{m,n=1}^{\infty} *(Ae_n, f_m)*^2 - \sum_{m,n=1}^{N} *(Ae_n, f_m)*^2 .$$

Which converges to O as $N \to \infty$, that it implies that A is compact.

Lemma 3. If $A = A^* \ge 0$ and $(e_n)_{n=1}^{\infty}$ is a complete orthonormal sets in , then its trace

$$tr*(A)* = \sum_{n=1}^{\infty} \langle Ae_n, e_m\rangle \in [0, +\infty]$$

does not depend upon the choice of $(e_n)_{n=1}^{\infty}$. If $tr*A* < \infty$ then A is compact. If $(\lambda_n)_{n=1}^{\infty}$ are its eigenvalues repeated according to their multiplicities, then

$$tr * A* = \sum_{n=1}^{\infty} \lambda_n .$$

Proof. Lemma 2 implies that $tr * A*$ does not depend on the choice of $\{e_n\}_{n=1}^{\infty}$, because

$$\sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle = \sum_{n=1}^{\infty} ** A^{1/2} e_n **^2 .$$

If the sum is finite then

$** A^{1/2} e_n **_2^2 = tr[A] < \infty$ .... (1)

So $A^{1/2}$ is compact by lemma 2. This implies that A is compact.

We say that a bounded operator A on a Hilbert space , is trace class if $tr[* A*] < \infty$. We will show that C_1 is a two-sided ideal in algebra $\mathrm{p}(,)$ of all bounded operators on ,.

Conclusion

Every finite rank of operators is trace class operators, every class operators is Hilbert-Schmidt operators, every Hilbert-Schmidt operators is compact operators, every compact operators is bounded operators, But converse is not necessarily true.

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