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Fixed Point Theorems For Generalized Contractions In Complete Metric Space

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Abstract:- In this paper , we present fixed point results for generalization on spaces with two metrics. The focus in on continuation results for such type of mappings.

Key words: - Metric Space, Complete Metric Space, Self Mapping, Fixed Point, Commute Mapping

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Introduction:-

The study of common fixed point of mapping contractive type condition has been a very active field of research activity during the last three decades. The most general of the common fixed point pertain to two or three mapping of a metric space (X, d) and use either a Banach type contractive condition or other contractive condition. Many, Hardy [1], Rajput [2] , Yadav [3] , Sengupta[4] and so many author work in this field and prove more interesting result. Throughout this section (X, d') denotes a complete metric space and d be an another metric on X. if $x_0 \in X$ and $r > 0$ denote by $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$ and by $\text{clos.} B(x_0, r)^{d'}$ the d'- closer of $B(x_0, r)$.

Fixed point results for Banach Generalized contractions:-

Theorem: 1

Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$ and T be the mapping from, $\text{clos.}B(x_0, r)^{d'}$ into X, satisfying the following conditions;

$$d(Tx, Ty) \leq \alpha . d(x, y) \tag{1.1}$$

Where non negative α, such that, $0 \leq \alpha < 1$

In addition assume the following three properties hold:

$$d(x_0, Tx_0) < (1 - \alpha)\, r \tag{1.2}$$

If $d \ngeq d'$

then T is uniformaly continuous from $(B(x_0, r), d)$ into (X, d') (1.3)

if $d \neq d'$ then T is continuous from $\left(\text{clos.}B(x_0, r)^{d'}, d'\right)$ into (X, d') (1.4)

then T has fixed point, that is there exists $x \in \text{clos.}B(x_0, r)^{d'}$ with $Tx = x$.

Proof:

Let $x_1 = Tx_0$ then from (1.2), we have

$$d(x_0, x_1) = d(x_0, Tx_0) < (1 - \alpha)\, r \leq r$$

So that, $x_1 \in B(x_0, r)$

Next let $x_2 = Tx_1$ then we note that,

$$d(x_1, x_2) = d(Tx_0, Tx_1)$$

From (1.1)

$$d(Tx_0, Tx_1) \leq \alpha\, d(x_0, x_1)$$

$$d(Tx_0, Tx_1) \leq \alpha\, (1 - \alpha)\, r$$

Now

$$d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$$

$$d(x_0, x_2) \leq (1 - \alpha)\, r + \alpha\, (1 - \alpha)\, r$$

$$d(x_0, x_2) \leq (1 - \alpha)\, r\, (1 + \alpha)$$

$$d(x_0, x_2) < (1 - \alpha)\, r\, (1 + \alpha + \alpha^2 + \alpha^3 + \cdots \dots \dots . .)$$

$$d(x_0, x_2) < (1 - \alpha)\, r\, (1 - \alpha)^{-1}$$

$$d(x_0, x_2) < r$$

So that, $x_2 \in B(x_0, r)$

Proceeding inductively we obtain

$$d(x_{n+1}, x_n) \leq \alpha^n d(x_0, x_1)$$

$$d(x_0, x_{n+1}) < (1-\alpha)^n r (1-\alpha)^{-1}$$

It follows $d(x_0, x_{n+1}) < r$ and $x_{n+1} \in B(x_0, r)$

In this way we construct a sequence $\{x_n\}$ of elements of X, such that $\{x_n\}$ is a Cauchy sequence with respect to, d , which converges to x.

We claim that $\{x_n\}$ is a Cauchy sequence with respect to d'.

If $d \geq d'$ then this is trivial.

Next we suppose that, $d \ngeq d'$

Let $\varepsilon > 0$ be given. Now from (1.3) that there exists $\delta > 0$ such that,

$d'(Tx, Ty) < \varepsilon$ whenever $x, y \in B(x_0, r)$ *and* $d(x, y) < \delta$ (1.5)

From the above the sequence $\{x_n\}$ is a Cauchy sequence with respect to d, so we know that there exists N with

$$d(x_n, x_m) < \delta \text{ for all } n, m \geq N \quad (1.6)$$

Now from (1.5) and (1.6) implies

$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon$ *whenever* $n, m \geq N$

Which proves that $\{x_n\}$ is a Cauchy sequence with respect to d'.

Now since (X, d') is complete there exists $x \in clos.B(x_0, r)^{d'}$ with

$d'(x_n, x) \to 0$ *and* $n \to \infty$.

We claim that, $x = Tx$ (1.7)

First consider the case, when $d \neq d'$.

$d'(x, Tx) \leq d(x, x_n) + d(x_n, Tx) = d(x, x_n) + d(Tx_{n-1}, Tx)$

Let $n \to \infty$ and using (1.4), we obtain

$d'(x, Tx) \leq d(x, x) + d(Tx, Tx)$

$d'(x, Tx) = 0$

And thus (8.7) is true,

Next we suppose that $d = d'$ then

$d'(x, Tx) \leq d(x, x_n) + d(Tx_{n-1}, Tx)$

From (1.1),

$$d'(x,Tx) \le d(x,x_n) + \alpha d(x_{n-1},Tx)$$

As $\to \infty$, $Tx_n = x = Tx$ and above inequality can be written as,

$$(1-\alpha)d(x,Tx) \le 0$$

So that, $, d(x,Tx) = 0$ and (1.7) holds.

This the proof of the theorem.

Theorem:- 2

Let (X,d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$ and T be the mapping from, $clos. B(x_0,r)^{d'}$ into X, satisfying the following conditions;

$$d(Tx,Ty) \le \alpha. d(x,y) + \beta[d(x,Tx) + d(y,Ty)] + \gamma[d(x,Ty) + d(y,Tx)] \quad (2.1)$$

Where non negative α, β, γ, such that, $0 \le \alpha + \beta + \gamma < 1$

In addition assume the following three properties hold:

$$d(x_0,Tx_0) < \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \quad (2.2)$$

If $d \ngeq d'$

then T is uniformaly continuous from $(B(x_0,r),d)$ *into* (X,d') (2.3)

if $d \neq d'$ then T is continuous from $\left(clos. B(x_0,r)^{d'}, d'\right)$ into (X,d') (2.4)

then T has fixed point, that is there exists $x \in clos. B(x_0,r)^{d'}$ with $Tx = x$.

Proof:

Let $x_1 = Tx_0$ then from (2.2), we have

$$d(x_0,x_1) = d(x_0,Tx_0) < \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \le r$$

So that, $x_1 \in B(x_0,r)$

Next let $x_2 = Tx_1$ then we note that,

$$d(x_1,x_2) = d(Tx_0,Tx_1)$$

From (2.1)

$$d(Tx_0,Tx_1) \le \alpha\, d(x_0,x_1) + \beta[d(x_0,x_1) + d(x_1,x_2)] + \gamma\, d(x_0,x_2)$$

$$d(Tx_0,Tx_1) \le \left(\frac{\alpha+\beta}{1-\beta-\gamma}\right)\left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r$$

Now

$$d(x_0, x_2) \le d(x_0, x_1) + d(x_1, x_2)$$

$$d(x_0, x_2) \le \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r + \left(\frac{\alpha+\beta}{1-\beta-\gamma}\right)\left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r$$

$$d(x_0, x_2) \le \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \left(1 + \frac{\alpha+\beta}{1-\beta-\gamma}\right)$$

$$d(x_0, x_2) < \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \left(1 + \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right] + \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^2 + \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^3 + \cdots \ldots \ldots\right)$$

$$d(x_0, x_2) < \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right) r \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^{-1}$$

$$d(x_0, x_2) < r$$

So that, $x_2 \in B(x_0, r)$

Proceeding inductively we obtain

$$d(x_{n+1}, x_n) \le \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^n d(x_0, x_1)$$

$$d(x_0, x_{n+1}) < \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^n r \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^{-1}$$

It follows $d(x_0, x_{n+1}) < r$ and $x_{n+1} \in B(x_0, r)$

In this way we construct a sequence $\{x_n\}$ of elements of X, such that $\{x_n\}$ is a Cauchy sequence with respect to, d , which converges to x.

We claim that $\{x_n\}$ is a Cauchy sequence with respect to d'.

If $d \ge d'$ then this is trivial.

Next we suppose that, $d >\ne d'$

Let $\varepsilon > 0$ be given. Now from (1.3) that there exists $\delta > 0$ such that,

$d'(Tx, Ty) < \varepsilon$ whenever $x, y \in B(x_0, r)$ *and* $d(x, y) < \delta$ (2.5)

From the above the sequence $\{x_n\}$ is a Cauchy sequence with respect to d, so we know that there exists N with

$$d(x_n, x_m) < \delta \ for\ all\ n, m \ge N \tag{2.6}$$

Now from (2.5) and (2.6) implies

$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon\ whenever\ n, m \ge N$

Which proves that $\{x_n\}$ is a Cauchy sequence with respect to d'.

Now since (X, d') is complete there exists $x \in clos.B(x_0, r)^{d'}$ with

$d'(x_n, x) \to 0$ *and* $n \to \infty$.

We claim that, $x = Tx$ (2.7)

First consider the case, when $d \neq d'$.

$d'(x,Tx) \leq d(x,x_n) + d(x_n,Tx) = d(x,x_n) + d(Tx_{n-1},Tx)$

Let $n \to \infty$ and using (2.4), we obtain

$d'(x,Tx) \leq d(x,x) + d(Tx,Tx)$

$d'(x,Tx) = 0$

And thus (2.7) is true,

Next we suppose that $d = d'$ then

$d'(x,Tx) \leq d(x,x_n) + d(Tx_{n-1},Tx)$

From (2.1),

$$d'(x,Tx) \leq d(x,x_n) + \alpha d(x_{n-1},Tx) + \beta[d(x_{n-1},Tx_{n-1}) + d(x,Tx)] + \gamma[d(x_{n-1},Tx) + d(x,Tx_{n-1})]$$

As $\to \infty$, $Tx_n = x = Tx$ and above inequality can be written as,

$$\left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right) d(x,Tx) \leq 0$$

So that, $, d(x,Tx) = 0$ and (2.7) holds.

This complete proof of the theorem.

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