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The Role Of Massive Neutrinos In Shaping Large-scale Structures Of The Universe: Analytical Derivations And Numerical Computations

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Abstract: The study of large-scale structures in the universe is fundamental to understanding cosmic evolution. In this paper, I explore the role of massive neutrinos in structure formation, with a particular focus on Jeans mass calculations and the implications for dark matter models. Using analytical derivations and computational analysis, I examine the Jeans wave number in different relativistic regimes and discuss the significance of fluctuations in the neutrino sphere. My findings highlight the impact of neutrino mass on the growth of cosmic structures and provide insights into the mixed dark matter model.

Keywords **-** Massive neutrinos, hot dark matter (HDM), cold dark matter (CDM), Jeans mass, large-scale structure formation, gravitational instability, structure growth, cosmic evolution

Introduction

The formation of cosmic structures remains one of the central problems in modern cosmology. The composition of dark matter plays a crucial role in determining the characteristics of these structures. Two primary models of dark matter have been proposed: hot dark matter (HDM) and cold dark matter (CDM). In this paper, I investigate the role of massive neutrinos, a leading HDM candidate, in structure formation by analyzing their effect on Jeans mass and wave numbers (Weinberg, 1972; Kolb & Turner, 1994).

Theoretical Background

Jeans mass calculations provide insight into the ability of a region of gas or particles to collapse under self-gravity. The Jeans wave number is derived from gravitational instability criteria, and its behavior in relativistic and non-relativistic regimes determines the formation of structures in the universe. Neutrinos, due to their relativistic nature in the early universe, contribute significantly to these calculations (Planck Collaboration, 2020).

Jeans Mass for Neutrinos

The mean occupation number of the neutrino background in the standard big-bang model as a function of momentum p and timed t is,

$$dn_v = \frac{g_v}{h^3}\frac{d^3p}{(e^{pc/KT}+1)} \tag{1}$$

which is valid even in the non-relativistic regime, where it is neither fermi Dirac nor degenerate.

Here, K is Boltzmann's constant, T is Temperature, and g_v is the spin-degeneracy equal to 2 for each species of Majorana neutrinos ($v - \bar{v}$ pair) and 4 for Dirac type.

$$\therefore \langle n \rangle = \int_0^\infty dn$$

$$= \int_0^\infty \frac{g_v}{(e^{pc/KT} + 1)} \cdot \frac{4\pi p^2 dp}{h^3} \qquad (2)$$

Substituting $\frac{pc}{KT} = y,$ we get,

$$\langle n \rangle = \frac{4\pi g_v}{h^3} \left(\frac{KT}{c}\right)^3 \int_0^3 \frac{y^2 dy}{(e^y + 1)}$$

$$= \frac{4\pi g_v}{h^3} \left(\frac{KT}{c}\right)^3 \Gamma(k)\, \eta(k) \qquad (3)$$

Where, $\eta(k)$ is the Reimann-Eta function and $\Gamma(k)$ is the Gamma function.

$$\text{Putting, } y = \frac{pc}{KT} = \frac{mc^2}{KT} = \left(\frac{1 + Z_{NR}}{1 + Z}\right)$$

Where, Z is red-shift and m is the rest mass of neutrino, we have

$$\langle n \rangle = \frac{8\pi g_v \eta(k)}{h^3} \left(\frac{mc}{x}\right)^3 \qquad (4)$$

Now, the general expression for energy is:

$$E = m_{\text{rel}} c^2 = (p^2 c^2 + m^2 c^4)^{1/2}$$

$$\text{or,} \quad m_{\text{rel}} = \frac{(p^2 c^2 + m^2 c^4)^{1/2}}{c^2} \qquad (5)$$

Also, the momentum is given by,

$$p = m_{\text{rel}} v = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

$$\text{or,} \quad v = \frac{pc^2}{(p^2 c^2 + m^2 c^4)^{1/2}} \qquad (6)$$

As derived earlier, the Jeans wave number is given by,

$$K_J = \left(\frac{4\pi G\rho}{v^2}\right)^{\frac{1}{2}}$$

$$= (4\pi G)^{\frac{1}{2}} \frac{[m_{rel} \langle n \rangle]^{\frac{1}{2}}}{V}$$

$$= [4\pi G \langle n \rangle]^{\frac{1}{2}} \cdot \frac{(p^2 c^2 + m^2 c^4)^{1/4}}{c} \cdot \frac{(p^2 c^2 + m^2 c^4)^{1/2}}{pc^2}$$

$$= \frac{[4\pi G \langle n \rangle]^{\frac{1}{2}}}{c^3} \cdot \frac{(p^2 c^2 + m^2 c^4)^{3/4}}{p} \qquad (7)$$

So, the expectation value of the Jeans wave number is given by,

$$\langle K_J \rangle = \frac{g_v}{\langle n \rangle} \int_0^\infty \frac{K_J}{(e^{pc/KT} + 1)} \frac{4\pi p^2}{h^3} dp$$

$$= \left(\frac{4\pi G}{\langle n\rangle}\right)^{1/2} \frac{4\pi g_v}{(ch)^3} \int_0^\infty \frac{p\, dp\, (p^2c^2 + m^2c^4)^{3/4}}{(e^{pc/KT} + 1)}$$

$$= \left(\frac{4\pi G}{\langle n\rangle}\right)^{3/2} \frac{4\pi g_v}{(ch)^3} \left(\frac{KT}{c}\right)^2 \frac{y\, dy\, (y^2K^2T^2 + m^2c^4)^{3/4}}{(e^y + 1)} \qquad \left[\because y = \frac{pc}{KT}\right]$$

$$= \left(\frac{2\pi}{ch}\right)^{3/2} \left(\frac{g_v G}{\pi\, \eta(k)}\right)^{1/2} \left(\frac{mc}{x}\right)^2 \int_0^\infty \frac{y\, dy\, (y^2 + x^2)^{3/4}}{(e^y + 1)} \tag{8}$$

$$\left[\because \text{Putting, } x = \frac{mc^2}{KT} \text{ and substituting for } \langle n\rangle \text{ from (4)}\right]$$

Since, $m_{pl.}$ = Planck mass = $\left(\frac{ch}{2\pi G}\right)^{1/2}$

$$= \frac{1}{\sqrt{G}} \quad \text{for} \quad \left[c = \frac{h}{2\pi} = 1\right], \quad \text{we have,}$$

$$\langle K_J\rangle = \frac{0.6\, {g_v}^{1/2}}{m_{pl}} \frac{m^2}{x^2} \int_0^\infty \frac{y\, dy\, (y^2 + x^2)^{3/4}}{(e^y + 1)}$$

$$= \frac{0.6\, {g_v}^{1/2}}{m_{pl}} \frac{m^2}{x^2} \int_0^\infty \left[\frac{y\, dy\, x^{3/2}}{(e^y + 1)} \left(1 + \frac{y^2}{x^2}\right)^{3/4}\right]$$

$$= \frac{0.6\, {g_v}^{1/2}}{m_{pl}} \frac{m^2}{x^2} \int_0^\infty \left[\frac{y\, dy\ x^{3/2}}{(e^y + 1)} \left\{1 + \frac{3}{4}\left(\frac{y^2}{x^2}\right) + \frac{\frac{3}{4}\left(\frac{3}{4} - 1\right)}{2!}\left(\frac{y^2}{x^2}\right)^2 + \cdots \ldots \ldots \ldots \ldots\right\}\right]$$

$$= \frac{0.6\, {g_v}^{1/2}}{m_{pl}} \frac{m^2}{x^2} \left[x^{3/2} \int_0^\infty \frac{ydy}{(e^y + 1)} + \frac{3}{4x^{1/2}} \int_0^\infty \frac{y^3dy}{(e^y + 1)} - \frac{3}{32x^{5/2}} \int_0^\infty \frac{y^5dy}{(e^y + 1)}\right]$$

[$\because$ Neglecting other higher order terms]

$$\therefore\ \langle K_J\rangle = \frac{0.6\, {g_v}^{1/2}\, m^2}{m_{pl}} \left[\frac{1}{x^{1/2}} \int_0^\infty \frac{ydy}{(e^y + 1)} + \frac{3}{4x^{5/2}} \int_0^\infty \frac{y^3dy}{(e^y + 1)} - \frac{3}{32x^{9/2}} \int_0^\infty \frac{y^5dy}{(e^y + 1)}\right] \tag{9}$$

Similarly,

$$\langle K_J^2\rangle = \frac{0.64\, g_v\, m^4}{m_{pl.}^2\, x^4} \int_0^\infty dy \frac{(y^2 + x^2)^{3/2}}{(e^y + 1)}$$

or, $$\langle K_J^2\rangle = \frac{0.64\, g_v\, m^4}{m_{pl}^2\ x^4} \int_0^\infty \left[\frac{dy\, x^3}{(e^y + 1)} \left(1 + \frac{y^2}{x^2}\right)^{3/2}\right]$$

$$= \frac{0.64\, g_v\, m^4}{m_{pl}^2\, x^4} \int_0^\infty \left[\frac{dy\, x^3}{(e^y + 1)} \left\{1 + \frac{3}{2}\left(\frac{y^2}{x^2}\right) + \frac{\frac{3}{2}\left(\frac{3}{2} - 1\right)}{2!}\left(\frac{y^2}{x^2}\right)^2 + \cdots \ldots \ldots \ldots \ldots \ldots\right\}\right]$$

$$= \frac{0.64\, g_\nu\, m^4}{m_{pl}^2\, x^4}\left[x^3\int_0^\infty \frac{dy}{(e^y+1)} + \frac{3x}{2}\int_0^\infty \frac{y^2 dy}{(e^y+1)} + \frac{3}{8x}\int_0^\infty \frac{y^4 dy}{(e^y+1)}\right]$$

[∵ Neglecting other higher order terms]

$$\therefore\ \langle K_J^2\rangle = \frac{0.64 g_\nu m^4}{m_{pl}^2}\left[\frac{1}{x}\int_0^\infty \frac{dy}{(e^y+1)} + \frac{3}{2x^3}\int_0^\infty \frac{y^2 dy}{(e^y+1)} + \frac{3}{8x^5}\int_0^\infty \frac{y^4 dy}{(e^y+1)}\right] \tag{10}$$

From equation (8), we see that $\langle K_J\rangle$ varies as x^{-2} i.e. as $(1+z)^2$ in extreme relativistic (ER) regime, where $y^2 \gg x^2$ and it varies as $x^{-1/2}$ i.e., as $(1+z)^{1/2}$ in non-relativistic (NR) regime. In the semi-relativistic regime, the integrals involved are to be calculated numerically. For a range of $x = 0$ to 10 , the values of $\langle K_J\rangle$ have been calculated using computer programs. The values of some required integrals have been shown in appendix.

We construct a table containing the values of Jeans wave numbers, $\langle K_J\rangle$ in the units of ${g_\nu}^{1/2} m^2/m_{pl}$ and their variances for different values of x.

Jeans mass calculations provide insight into the ability of a region of gas or particles to collapse under self-gravity. The Jeans wave number is derived from gravitational instability criteria, and its behavior in relativistic and non-relativistic regimes determines the formation of structures in the universe. Neutrinos, due to their relativistic nature in the early universe, contribute significantly to these calculations (Planck Collaboration, 2020).

Methodology

I employ analytical derivations and numerical computations to evaluate the Jeans wave number. This involves calculating the mean occupation number of the neutrino background, integrating over momentum distributions, and considering dependencies on temperature and redshift factors. The computational approach includes numerical solutions for Jeans wave numbers across a range of relativistic to non-relativistic transitions.

Results and Discussion

Tables and figures are integral to illustrating the computational results. The following sections present detailed tables and graphical representations of key findings:

Table 1 Numerical results for Jeans wave number

S. No.	$x = \frac{mc^2}{KT}$	Jeans Wave Number $\langle K_J\rangle$	$\langle K_J^2\rangle$	$\langle K_J\rangle^2$	$\Delta K_J = \left[\langle K_J^2\rangle - \langle K_J\rangle^2\right]^{1/2}$
1.	0.5	7.6878024	61.6004505	59.1023059	1.580
2.	1.0	2.1218862	4.6126758	4.50240105	0.332
3.	1.5	1.078892	1.2191037	1.16400795	0.234
4.	1.6	1.0132675	1.1155140	1.02671103	0.298
5.	1.7	0.9837282	1.1914502	0.96772118	0.473
6.	1.8	0.723568	0.762673	0.52355065	0.489
7.	1.9	0.6575213	0.7037756	0.43233426	0.521
8.	2.0	0.506950	0.61316159	0.256998302	0.596
9.	2.1	0.5045928	0.5352585	0.25461389	0.529
10.	2.2	0.4974203	0.4728567	0.247426956	0.474
11.	2.3	0.4873574	0.4221+20	0.237510724	0.429

12.	2.4	0.47565815	0.3803761	0.226250675	0.392
13.	2.5	0.4631418	0.3455665	0.21450039	0.362
14.	3.0	0.401524	0.2350246	0.16122156	0.271
15.	3.5	0.367451	0.1777806	0.13502039	0.206
16.	4.0	0.3136534	0.1434183	0.09837846	0.212
17.	4.5	0.2845715	0.1206111	0.08098099	0.199
18.	5.0	0.2622770	0.1043625	0.06878596	0.188
19.	5.5	0.2433645	0.092174	0.05922629	0.181
20.	6.0	0.2283643	0.0826696	0.05215029	0.171
21.	6.5	0.215835	0.07503398	0.04658512	0.168
22.	7	0.2051948	0.0687532	0.0421049	0.163
23.	7.5	0.196025	0.06348756	0.03844256	0.158
24.	8.0	0.1880235	0.05900347	0.03535284	0.153
25.	8.5	0.809647	0.05513470	0.03274825	0.149
26.	9.0	0.1746786	0.0517598	0.03051262	0.145
27.	9.5	0.1690339	0.04878393	0.02857246	0.142
28.	10.0	0.1639282	0.04614839	0.02687250	0.138

Now, a graph is plotted between ΔK_J versus x.
Where,
$\Delta K_J =$ Deviation between $\langle K_J^2 \rangle$ and $\langle K_J \rangle^2$ under square root.
$x =$ Reciprocal of red-shifts

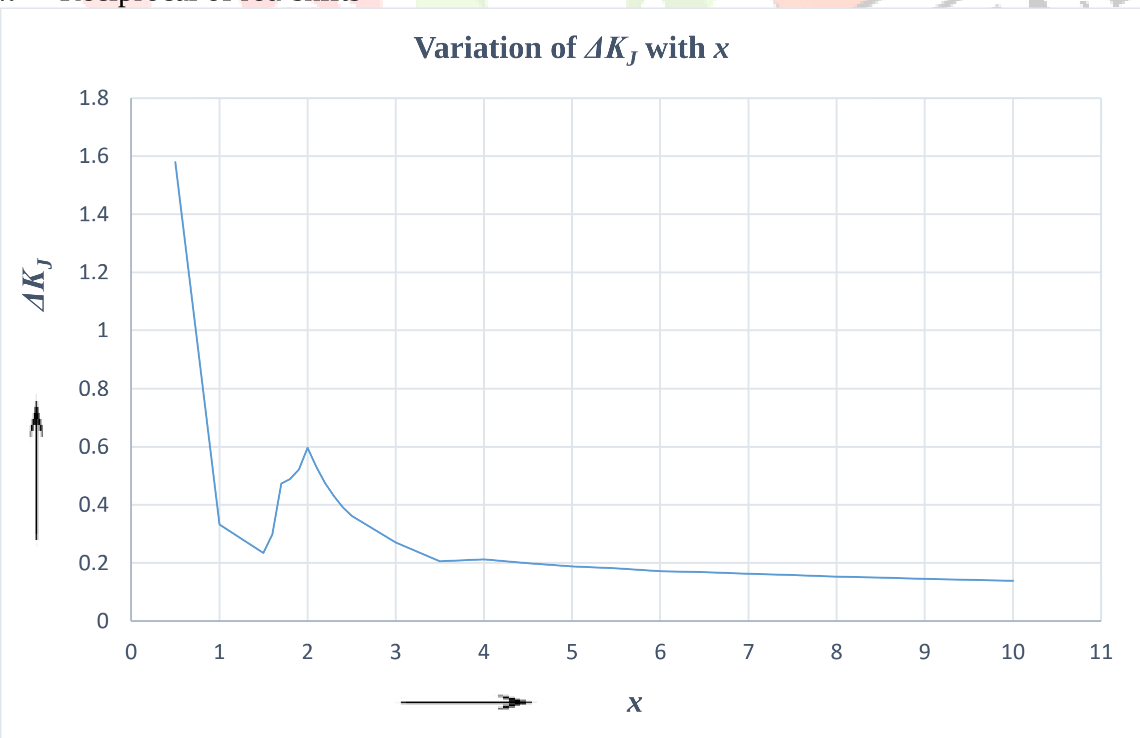

Figure 1 Graphical representation of ΔK_J variations. The plot illustrates the variation of ΔK_J with x. It is observed that for x in the range of 1.5 to 2.5, ΔK_J exhibits fluctuations, while beyond this range, it remains nearly constant.

My results demonstrate a clear dependence of Jeans wave number on the neutrino mass and redshift. Key observations include:

1. The Jeans wave number varies as x^{-2} in the extreme relativistic regime and $x^{-1/2}$ in the non-relativistic regime.
2. A significant fluctuation in ΔK_J is observed in the transition range $x \sim$ (1.5 to 2.5) indicating a broad spectrum of neutrino spheres. So a wide spectrum of the neutrino spheres may be expected to develop at this time. Hence, a variation of the size of the neutrino sphere dating from this time is possible.
3. Mixed dark matter (MDM) models incorporating both Hot dark matter (HDM) and Cold dark matter (CDM) elements offer a more refined understanding of structure formation (Riess et al., 2021; SDSS Collaboration, 2019).

Conclusion

Massive neutrinos play a critical role in cosmic structure formation, influencing the Jeans mass and wave number in different relativistic phases. The transition from relativistic to non-relativistic states leads to observable fluctuations that impact the growth of cosmic structures. These findings support the necessity of incorporating neutrino effects into cosmological simulations and dark matter models (The Millennium Simulation Project, 2005).

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Appendix

1. $Integral\ 1\ (I_1) = \int_0^\infty \frac{dy}{(e^y+1)} = 0.69315$
2. $Integral\ 2\ (I_2) = \int_0^\infty \frac{y\,dy}{(e^y+1)} = 0.82247$
3. $Integral\ 3\ (I_3) = \int_0^\infty \frac{y^2\,dy}{(e^y+1)} = 1.803$
4. $Integral\ 4\ (I_4) = \int_0^\infty \frac{y^3\,dy}{(e^y+1)} = 5.6822$
5. $Integral\ 5\ (I_5) = \int_0^\infty \frac{y^4\,dy}{(e^y+1)} = 23.331$
6. $Integral\ 6\ (I_6) = \int_0^\infty \frac{y^5\,dy}{(e^y+1)} = 118.27$