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Gauge Modules Over The Lie Algebra Of Vector Fields

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Abstract

In this paper, we investigate a specific category of gauge modules on a smooth irreducible affine algebraic variety. These modules admit compatible actions by both the algebra A of regular functions and the Lie algebra V of vector fields on the variety. Our main result establishes that a gauge module, associated with a simple gl_N-module, maintains its irreducibility when considered as a module over the Lie algebra of vector fields.

Keywords : Lie algebra, Gauge Module, Dimensions of algebra

1.0 Introduction

Affine algebraic varieties and the corresponding structures of function algebras and vector fields provide a rich framework for exploring the interplay between geometry and algebra. The concept of gauge modules, which inherently interact with both function algebras and Lie algebras of vector fields, offers a fertile ground for such studies. The notion of gauge modules aligns with various applications in representation theory, mathematical physics, and differential geometry.

In this work, we focus on smooth irreducible affine algebraic varieties and study gauge modules that support actions from both the algebra A of functions and the Lie algebra V of vector fields. We aim to understand the conditions under which these gauge modules, especially those corresponding to simple gl_N-modules, remain irreducible as modules over V.

Lie algebras of vector fields on smooth irreducible affine varieties exhibit unique structural characteristics that distinguish them from simple finite-dimensional Lie algebras. Notably, as demonstrated in [3], these algebras often lack non-zero nilpotent or semisimple elements, rendering traditional tools like roots and weights inapplicable for their representation theory. The development of this field began with the pioneering works [4] and [5], which introduced the study of a category of finite rank AV-modules. These modules admit compatible actions from both the Lie algebra V of vector fields and the commutative algebra A of polynomial functions on the variety, and are finitely generated as A-modules.

Inspired by non-abelian gauge theory, [4] introduced the concept of gauge modules, a special family of finite rank AV-modules. The construction of a gauge module involves two key components:

1. A finite-dimensional representation U of a Lie algebra L+, a subalgebra generated by elements of non-negative degrees in Der(K[t1,…,tN]).
2. Gauge fields {Bi}

The main result from [4] established that if U is an irreducible finite-dimensional gl_N-module, then the corresponding gauge module is also irreducible as an AV-module.

2.0 Objective

This paper aims to further explore the irreducibility of simple gauge AV-modules, particularly as modules over the Lie algebra V of vector fields. We prove that a gauge AV-module associated with an irreducible finite-dimensional gl_N-module U remains irreducible as a V-module, except when U is an exterior power of the natural N-dimensional gl_N-module. These exceptional AV-modules manifest within the de Rham complex.

3.0 Key Concepts and Techniques

One of the critical insights is that the de Rham differential is a homomorphism of V-modules (but not AV-modules). Consequently, the kernels and images of the de Rham differential are V-submodules within the corresponding gauge modules. This generalizes a theorem by Eswara Rao [7] on irreducible tensor modules over the Lie algebra of vector fields on a torus.

Our proof strategy involves recovering the A-action from the V-action, aiming to show that every V-submodule in a gauge module is also an AV-submodule. This approach fails precisely in the case of de Rham modules. A key technical tool we employ is Hilbert's Nullstellensatz.

4.0 Definitions

Let K be an algebraically closed field of characteristic 0, and let X⊂AKn be a smooth irreducible affine variety of dimension N.

4.1 Algebra of Polynomial Functions

The ideal I=⟨g1,…,gm⟩ in K[x1,…,xn] consists of the polynomial functions that vanish on X. The algebra A=K[x1,…,xn]/I represents the polynomial functions on X.

4.2 Lie Algebra of Vector Fields

The Lie algebra V=Der(A) consists of all derivations of the algebra A. These derivations are the vector fields on X.

4.3 AV-Modules

An AV-module is a vector space M that supports structures of both a module over the commutative unital algebra A and a module over the Lie algebra V, such that these two module structures are compatible. This compatibility is expressed by the following condition: η·(f·m)=η(f)·m+f·(η·m), for all ∈η∈V, ∈f∈A, and ∈m∈M.

4.4 Finite Rank AV-Modules

An AV-module M is said to have finite rank if it is finitely generated as an A-module.

4.5 Jacobian Matrix and Rank

Let J=(∂xi∂gj) be the Jacobian matrix of the ideal generators g1,…,gm. The rank of J over the field F of rational functions on X is denoted by r. Given this rank, the dimension of X is N=dimX=n−r.

4.6 Non-zero Minors and Smoothness Criterion

Let {hi} be the set of non-zero r×r minors of J. For each minor hi, the set N(hi) is defined as: N(hi)={p∈X∣hi (p)=0}. The Jacobian criterion for smoothness states that: ⋃iN(hi)=X. This means that the variety X is covered by the open sets where each r×r minor is non-zero.

5.0 Lie Algebra V and its Description

The Lie algebra V of vector fields can be described as an A-submodule of the free A-module ⨁i=1nA∂xi∂. Specifically, an element ∑i=1nfi∂xi∂ belongs to V if and only if: ∑i=1nfi∂xi∂gj=0 in A for all j=1,…,m. This condition ensures that the vector fields respect the relations defining X (see [3]).

5.1 Constructing Derivations Locally

Fix a non-zero minor *h*h of J, and let β⊂{1,…,n} be the set of columns of J involved in *h*h. Since the rank of J is r=∣β∣, we can solve the system of linear equations ∑i=1nfi∂xi∂gj=0 over F by treating fi for i∈/β as free variables. For each i∈/β, we construct the derivations: τi=∂xi∂+∑j∈βfij∂xj∂, where ∈*h*−1 fij∈h−1A. Consequently, hτi∈Der(A). It is important to note that while each τi is a derivation of the localized algebra A(h), it is not necessarily a derivation of A.

5.2 Definition from [5]

We now present a key definition from [5]:

Definition: An AV-module M is a vector space equipped with module structures over both the commutative unital algebra A and the Lie algebra V of vector fields, such that these structures are compatible. This compatibility is given by: η·(f·m)=η(f)·m+f·(η·m), for all ∈η∈V, ∈f∈A, and ∈m∈M.

5.3 Key Points

1. Algebraically Closed Field K: We work over an algebraically closed field of characteristic 0, ensuring rich algebraic properties and the applicability of various algebraic theorems.
2. Affine Variety X: The smooth irreducible affine variety X is embedded in the affine space AKn with dimension N.
3. Ideal I: The ideal I in K[x1,…,xn] captures the polynomial functions that vanish on X.
4. Algebra A: The algebra A=K[x1,…,xn]/I represents the regular functions on X.
5. Lie Algebra V: The Lie algebra V=Der(A) consists of vector fields on X, acting as derivations on the algebra A.
6. AV-Module Compatibility: An AV-module M must satisfy the compatibility condition, ensuring that the actions of A and V on M are coherently linked.
7. Finite Rank: The notion of finite rank for an AV-module implies that M is finitely generated over the algebra A, ensuring a manageable and structured module theory.

6.0 Gauge Modules

Let us delve into the family of gauge AV-modules introduced in [4]. We start by considering the Lie algebra L=Der(K[t1,…,tN]), which is the Lie algebra of derivations on the polynomial ring K[t1,…,tN].

6.1 Z-Grading of L

The Lie algebra L possesses a natural Z-grading: L=L−1⊕L0⊕L1⊕L2⊕⋯. Here, L−1 consists of derivations decreasing the degree of polynomials by 1, L0 consists of degree-preserving derivations, and Lk (for k≥1) consists of derivations increasing the degree of polynomials by k.

6.2 Subalgebra L+

The subalgebra L+ is defined as the subalgebra of L consisting of elements of non-negative degree: L+=L0 ⊕L1⊕L2⊕⋯. Note that L0 is spanned by the elements ti∂tj∂ and is isomorphic to gl_N.

6.3 **Definition of Gauge Fields**

Let (U,ρ) be a finite-dimensional L+-module. The functions ⎕:(ℎ)⊗→(ℎ)⊗Bi:A(h)⊗U→A(h)⊗U for i=1,…,N are called gauge fields if they satisfy the following conditions:

1. (ℎ)A(h)-Linearity: Each Bi is (ℎ)A(h)-linear.
2. Commutativity with ρ(L+):[Bi,ρ(L+)]=0.
3. Commutativity of Derived Operators: [∂ti∂+Bi,∂tj∂+Bj]=0 as operators on (ℎ)⊗A(h)⊗U for all i,j∈{1,…,N}.

Action on A(h)⨂U

Given the gauge fields {Bi} for =1,…,i=1,…,N, the space (ℎ)⨂A(h)⨂U becomes a Der((ℎ))Der(A(h))-module with the following action: f∂ti∂·(g⨂u)=f∂ti∂g⨂u+fg⨂Biu+∑k∈ZN+k!1g∂tk∂kf⨂ρ(tk∂ti∂)u, where (ℎ)f,g∈A(h) and ∈u∈U.

6.4 Interpretation

1. First Term: f∂ti∂g⨂u corresponds to the standard action of a derivation on a product of functions.
2. Second Term: fg⨂Biu involves the action of the gauge field Bi on u, scaled by the function g.
3. Sum Term: The term ∑k∈ZN+\{0}k!1g∂tk∂kf⨂ρ(tk∂ti∂)u accounts for higher-order interactions between the derivation and the module action, involving derivatives of f and the representation ρ.

Gauge modules are constructed by combining finite-dimensional representations of a subalgebra L+ of derivations with additional functions called gauge fields. These fields ensure the compatibility and integrability of the module actions, leading to a structured module over the localized algebra (ℎ)A(h). This construction allows for the study of gauge modules' irreducibility and their interaction with the Lie algebra of vector fields on affine varieties.

Let X be a smooth irreducible affine algebraic variety, and let M be a gauge module corresponding to a simple finite-dimensional glN-module U. Then M is a simple AV-module.

6.5 Assumptions

For the rest of the paper, we will assume that M is a gauge module corresponding to a simple finite-dimensional gl_N-module U.

Standard Generators of the Center ofU(glN)

We shall fix the standard generators of the center of the universal enveloping algebra U(glN). These are given by the Casimir elements: Ωk=∑i1,…,ik=1NEi1i2Ei2i3⋯Eiki1, where Eij are the standard matrix units of gl_N. The center Z(glN) is then generated by these elements: Z(gl_N)=K[Ω1,…,ΩN].

Investigation Strategy

The strategy involves analyzing the action of V on the gauge module M and reconstructing the A-action from this V-action. This approach aims to demonstrate that any V-submodule of M is also an AV-submodule, thereby proving the irreducibility of M as a V-module, except in the exceptional cases involving exterior powers of the natural gl_N-module.

1. Basics of Lie Algebras of Vector Fields and AV-Modules: We recall the fundamental concepts and definitions related to Lie algebras of vector fields on affine varieties and the category of AV-modules.
2. Construction of Gauge Modules: We discuss the construction of gauge modules, particularly focusing on those corresponding to simple gl_N-modules, and the role of gauge fields in this construction.
3. Irreducibility Over V: We delve into the conditions under which gauge modules remain irreducible over the Lie algebra V of vector fields, specifically identifying the exceptional cases.

4. de Rham Complex: We examine the connection between gauge modules and the de Rham complex, highlighting the role of exterior powers of the natural gl_N-module.
5. Connections to sl_2-Modules: We explore the structure of gauge modules in the context of sl_2-modules, providing an example with the Lie algebra W1 of vector fields on a circle.

Lemma 6 and Its Proof

If a gauge module M is reducible as a V-module, then Ω1Ω1 acts on U by a scalar from the set {0,1,…,N}.

Proof

Let ′M′ be a nontrivial V-submodule in M. We consider one of the charts (ℎ)N(h) of X with its chart parameters. For f∈A, i∈{1,2,…,N}, c∈k, and ≤20≤r≤2, we examine the composition of the actions of vector fields from V on (ℎ)⨂A(h)⨂U:

((ti+c)2−rfh∂ti∂)∘((ti+c)rh∂ti∂)(g⨂u).

When expanded using the given action, we obtain a quadratic polynomial in r. Using a Vandermonde determinant argument, we deduce that ′M′ is invariant under the terms corresponding to each power of r.

Quadratic Term Analysis

The operator corresponding to the −r2 term is:

ti+c)2fh2(Eii2−Eii).

Since c is arbitrary, we conclude that ′M′ is invariant under:

fh2(Eii2−Eii).

Hilbert's Nullstellensatz Application

Given that X is smooth, the set of functions {hj} determining the charts of X have no common zeros on X. By Hilbert's Nullstellensatz, the ideal generated by {hj2} contains 1. Therefore, ′M′ is invariant under Eii2−Eii, and more generally f(Eii2−Eii) for any f∈A.

Eigenspace Decomposition

Consider the decomposition of U into the joint eigenspaces for the family of commuting diagonalizable operators Eii2−Eii, i=1,…,N. For m∈M′, expand m in the joint eigenvectors for Eii2−Eii:

m=∑αmα.

By a Vandermonde argument, each component mα is in ′M′.

Zero Eigenspace Consideration

Assume that the zero eigenspace is trivial. This means that for each component mα, there exists at least one i such that Eii2−Eii acts by a non-zero scalar. Therefore, for every function f∈A, fmα∈M′. This implies that fm∈M′, making ′M′ an AV-submodule of M.

Contradiction and Conclusion

However, Theorem states that M is irreducible as an AV-module, which leads to a contradiction. Thus, the zero eigenspace must be nontrivial. This means that each Eii acts on this space as either 0 or 1. Consequently, Ω1=∑i=1NEii acts on U by a scalar from the set {0,1,…,N}

Theorem: Let M be a gauge module corresponding to a simple gl_N-module on a smooth irreducible affine algebraic variety. Then M is irreducible as a module over the Lie algebra of vector fields V on the variety, except when M appears in the de Rham complex.

This paper contributes to the theory of gauge modules by establishing a clear criterion for their irreducibility in the context of vector fields on smooth irreducible affine algebraic varieties. Future work could extend these results to more general settings or explore the deeper geometric and physical interpretations of the identified exceptional cases.

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