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Cg-Closed Sets And C-Normal Spaces

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Abstract: In this paper, we introduced a new class of sets called C-generalized closed (briefly Cg-closed) set which is a simultaneous generalization of C-closed and g-closed sets. First we investigated basic some properties of Cg-closed sets and then we obtained the relationship of Cg-closed sets with some other existing generalized closed sets. Moreover, we introduced the notion of C-Normal space by using C-closed sets, also we obtained some basic characterizations, properties and preservation theorems of C-normal spaces. Further, we also introduced some function related to Cg-open sets and investigated their properties with C-normal spaces.

Keyword:- C-closed set, F-closed sets, Cg-closed set, Fg-closed set, C-normal Space, almost Cg-closed function, almost Cg-continuous function etc.

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1. Introduction

Closed sets play a major role in the study of topological spaces. Generalized closed sets are a very useful research topic in topological spaces for many Topologists. In 1923, Tietze **[8]** first introduced the concept of normal spaces and studied their properties. In 1937, M. Stone **[7]** introduced the notion of regular open sets. In 1963, N. Levine **[4]** defined the concept of semi open sets and investigated their properties. In 1970, N. Levine **[5]** introduced the notion of generalized closed sets and studied the properties of g-closed sets in topological spaces. In 2002, K. Chandrasekhara Rao and K. Joseph **[6]** introduced the concept of s*g-closed sets in topological spaces. In 2023, Mesfer H. Alqahtani **[1]** introduced the concept of F-open and F-closed sets in topological spaces. In 2023, Mesfer H. Alqahtani **[2]** introduced the concept of C-open sets in topological spaces. In 2024, Hamant Kumar, B. S. Sharma and Anuj Kumar **[3]** introduced the concept of Fg-closed set which is the generalization of F-closed sets. They studied basic properties of these sets and examine the relationships between Fg-open and Fg-closed sets with other kinds of closed and open sets such as semi open, semi closed, w-open, w closed and g-open and g-closed sets etc.

2. Preliminaries

Throughout in this paper, spaces (X, ℑ), (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let f: X→Y (or simply f) always denote a mapping from space X to space Y. Let B be a subset of a space X. The closure of B, interior of B and complement of B is denoted by cl(B), int(B) and B^c (or X – B) respectively.

Definition 2.1: A subset B of a topological space (X, ℑ) is said to be:

(i) **regular open [7]** if B = int(cl(B)).
(ii) **semi open [4]** if B ⊂ cl(int(B)).
(iii) **F-open [1]** if cl(B) – B is finite set and B is open in X.
(iv) **C-open [2]** if cl(B) – B is countable set and B is open in X.

The complement of a regular open (resp. semi open, F-open and C-open set) set is called **regular closed** (resp. **semi closed, F-closed** and **C-closed**) set.

The intersection of all regular closed (resp. semi closed, F-closed and C-closed) sets containing B, is called **regular closure** (resp. **semi closure, F-closure** and **C-closure**) of B, and is denoted by **r-cl(B)** (resp. **s-cl(B), F-cl(B)** and **C-cl(B)**). The union of all regular open (resp. semi open, F-open and C-open) sets contained in B, is called **regular interior** (resp. **semi interior, F-interior** and **C-interior**) of B, and is denoted by **r-int(B)** (resp. **s-int(B), F-int(B)** and **C-int(B)**).
The collection of all regular open (resp. semi open, F-open and C-open) sets in X is denoted by **r-O(X)** (resp. **s-O(X)**, **F-O(X)** and **C-O(X)**). The collection of all regular closed (resp. semi closed, F-closed and C-closed) sets in X is denoted by **r-C(X)** (resp. **s-C(X)**, **F-C(X)** and **C-C(X)**).

Remark 2.2 From the above definitions the relationship among C-open sets and some other existing weaker and stronger forms of open sets are given as:

F-open → C-open → open → semi open

Where none of the implications is reversible can be seen from the following examples:

Example 2.3 Let X = {a, b, c} and ℑ = {ϕ, {a}, X}. Then {a, b} is semi open set in X but not open set in X.

Example 2.4 Let (ℝ, **U**) be the usual topological space then interval [2, 5) is semi open in ℝ as [2, 5) ⊂ cl(int([2, 5))) but not open in ℝ .

Example 2.5 Let X = ℝ and ℑ is the collection of all those subsets of ℝ which do not contain any irrational numbers together with ℝ then (ℝ, ℑ) be a topological space. Now the set of rational number ℚ be an open set in (ℝ, ℑ) but not a C-open set in (ℝ, ℑ) as: cl(ℚ) – ℚ = ℝ – ℚ = $ℚ^C$ *(set of irrational numbers)* which is an uncountable set.

Example 2.6 Let X = ℝ and ℑ is the collection of all those subsets of ℝ which contains a particular point 0 together with empty set ϕ then (ℝ, ℑ) be a topological space. Now the set of integer ℤ be an open set in (ℝ, ℑ) but not C-open set in (ℝ, ℑ) as: cl(ℤ) – ℤ = ℝ – ℤ which is not a countable set.

Example 2.7 The set of natural numbers ℕ is a closed set of usual topological spaces (ℝ, **U**) then ℝ – ℕ is open set in ℝ, also C-open set in ℝ but not F-open set in ℝ as: cl(ℝ – ℕ) – (ℝ – ℕ) = ℝ – (ℝ – ℕ) = ℕ which is countable set but not finite set.

Definition 2.8 A subset B of a topological space (X, ℑ) is said to be:

(i) **g-closed [5]** if cl(B) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(ii) **s*g-closed [6]** if cl(B) ⊂ U whenever A ⊂ U and U is semi-open.
(iii)**Fg-closed [3]** if cl(B) ⊂ U whenever A ⊂ U and U is F-open.

3. Cg-closed sets

Definition 3.1 A subset B of a topological space (X, ℑ) is said to be **Cg-closed** if cl(B) ⊂ U whenever B ⊂ U and U is C-open. The complement of the Cg-closed set is called Cg-open set. The collection of all Cg-open (resp. Cg-closed) sets is denoted by Cg-O(X) (resp. Cg-C(X)).

The intersection of all Cg-closed sets containing B, is called the Cg-closure of B and is denoted by Cg-cl(B). The Cg-interior of B, denoted by Cg-int(B) is defined to be the union of all Cg-open sets contained in B.

Theorem 3.2 Every s*g-closed set is Cg-closed set.
Proof: Let B be an s*g-closed set in X and let B ⊂ U where U is C-open in X. Now every C-open set is semi open set and B is s*g-closed, so by the definition of s*g-closed set, cl(B) ⊂ U, hence B is Cg-closed set in X.

Theorem 3.3 Every g-closed set is Cg-closed set.
Proof: Let B be a g-closed set in X and let B ⊂ U where U is C-open in X. Now every C-open set is open set and B is g-closed, so by the definition of g-closed set, cl(B) ⊂ U, hence B is Cg-closed set in X.

Theorem 3.4 Every Cg-closed set is Fg-closed set.
Proof: Let B be a Cg-closed set in X and let B ⊂ U where U is F-open in X. Now every F-open set is C-open set and B is Cg-closed, so by the definition of Cg-closed set, cl(B) ⊂ U, hence it is clear that B is Fg-closed set in X.

Remark 3.5 We summarize the fundamental relationships between several types of generalized closed sets by the following implications:

closed → s*g-closed → g-closed → Cg-closed → Fg-closed

The converse of the above implication may not be true as can be seen from the following examples:

Example 3.6 Let for the set of real numbers ℝ, the collection of open sets ℑ = {ϕ, ℚ, $ℚ^C$, ℝ} then (ℝ, ℑ) be a topological space. The set of integer ℤ is not closed in (ℝ, ℑ) as cl(ℤ) = ℚ, but ℤ is an s*g-closed set as ℚ is the smallest semi open set which contains ℤ and cl(ℤ) = ℚ ⊂ ℚ.

Example 3.7 For the set of real numbers ℝ, let the collection of open sets ℑ = {ϕ, ℕ, ℝ} *(where ℕ is the set of natural number)* then (ℝ, ℑ) be a topological space. Now the set of integer ℤ is a g-closed in (ℝ, ℑ) as: ℝ is the smallest open set which contains ℤ *(because ℤ is not open)* and cl(ℤ) = ℝ also contained in ℝ. But ℤ is not s*g-closed set in (ℝ, ℑ) as: ℤ be a semi open in (ℝ, ℑ) *(because ℤ ⊂ cl(int(ℤ)) = cl(ℕ) = ℝ)* and ℤ ⊂ ℤ but cl(ℤ) = ℝ is not subset of ℤ.

Example 3.8 By **example 2.5** the set of rational numbers ℚ is a Cg-closed set in (ℝ, ℑ) as set of real numbers ℝ is the smallest C-open set containing ℚ *(because ℚ is not C-open set in (ℝ, ℑ))* and cl(ℚ) = ℝ ⊂ ℝ. But ℚ is not a g-closed set in (ℝ, ℑ) as ℚ is open in (ℝ, ℑ) also ℚ ⊂ ℚ but cl(ℚ) = ℝ is not a subset of ℚ.

Example 3.9 For topological spaces $(\mathbb{R}, \Im)$, where $\Im = \{\phi, \mathbb{N}, \mathbb{R}\}$. Now the set of natural numbers $\mathbb{N}$ is a Cg-closed set in $(\mathbb{R}, \Im)$ as the set of real numbers $\mathbb{R}$ is the smallest C-open set containing $\mathbb{N}$ *(because $\mathbb{N}$ is not C-open set in $(\mathbb{R}, \Im)$ as $\mathbb{N}$ is open and $cl(\mathbb{N}) - \mathbb{N} = \mathbb{R} - \mathbb{N}$ which is an uncountable set)* and $cl(\mathbb{N}) = \mathbb{R} \subset \mathbb{R}$. But $\mathbb{N}$ is not a g-closed set in $(\mathbb{R}, \Im)$ as $\mathbb{N}$ is an open set in $(\mathbb{R}, \Im)$ also $\mathbb{N} \subset \mathbb{N}$ but $cl(\mathbb{N}) = \mathbb{R}$ is not a subset of $\mathbb{N}$.

Example 3.10 For the topological space $(\mathbb{R}, \Im)$ where $\mathbb{R}$ is the set of real numbers and $\Im$ be the collection of open sets and $\Im = \{\phi, \mathbb{Q}^C, \mathbb{R}\}$, the set of irrational number $\mathbb{Q}^C$ is a C-open set in $(\mathbb{R}, \Im)$ as $\mathbb{Q}^C$ is open set in $(\mathbb{R}, \Im)$ and $cl(\mathbb{Q}^C) - \mathbb{Q}^C = \mathbb{R} - \mathbb{Q}^C = \mathbb{Q}$ which is a countable set. Now $\mathbb{Q}^C$ is not a Cg-closed set in $(\mathbb{R}, \Im)$ as $\mathbb{Q}^C$ is C-open set and $\mathbb{Q}^C \subset \mathbb{Q}^C$ but $cl(\mathbb{Q}^C) = \mathbb{R}$ is not a subset of $\mathbb{Q}^C$, but $\mathbb{Q}^C$ is an Fg-closed set in $(\mathbb{R}, \Im)$ as $\mathbb{R}$ is the smallest F-open set which contains $\mathbb{Q}^C$ *(because $\mathbb{Q}^C$ is not F-open in $(\mathbb{R}, \Im)$)* and $cl(\mathbb{Q}^C) = \mathbb{R} \subset \mathbb{R}$.

4. Properties of Cg-closed sets

Theorem 4.1: Union of two Cg-closed set is Cg-closed set.
Proof: Let J and K be two Cg-closed sets. Let U be a C-open set containing $J \cup K$. Now J is Cg-closed set then $cl(J) \subset U$ as $J \subset U$ and U is C-open set, also K is Cg-closed set then $cl(K) \subset U$ as $K \subset U$ and U is C-open set. Now $cl(J) \subset U$ and $cl(K) \subset U \Rightarrow cl(J) \cup cl(K) \subset U \Rightarrow cl(J \cup K) \subset U$ *(because $cl(J \cup K) = cl(J) \cup cl(K)$)*. Hence $cl(J \cup K) \subset U$ whenever $J \cup K \subset U$ and U is C-open set. Hence $J \cup K$ is Cg-closed set.

In general finite union of Cg-closed sets is Cg-closed set.

Theorem 4.2: Intersection of two Cg-closed set is Cg-closed set.
Proof: Let J and K be two Cg-closed set. Now J is Cg-closed set if $cl(J) \subset U_1$ whenever $J \subset U_1$ and U_1 is C-open set, also K is Cg-closed set if $cl(K) \subset U_2$ whenever $K \subset U_2$ and U_2 is C-open set. Now $U_1 \cap U_2$ is C-open set as U_1 and U_2 are C-open sets, and $J \cap K \subset U_1 \cap U_2$ as $J \subset U_1$ and $K \subset U_2$. Now $cl(J) \subset U_1$ and $cl(K) \subset U_2 \Rightarrow cl(J) \cap cl(K) \subset U_1 \cap U_2 \Rightarrow cl(J \cap K) \subset U_1 \cap U_2$ *(because $cl(J \cap K) \subset cl(J) \cap cl(K)$)*. Hence $cl(J \cap K) \subset U_1 \cap U_2$ whenever $J \cap K \subset U_1 \cap U_2$ and $U_1 \cap U_2$ is C-open set. Hence $J \cap K$ is Cg-closed set.

In general finite intersection of Cg-closed sets is Cg-closed set.

Theorem 4.3: Union of two Cg-open sets is Cg-open set.
Proof: Let G and H be two Cg-open subset of a topological space $(X, \Im)$. Then $X - G$ and $X - H$ be two closed Cg-subset of X. Hence $(X - G) \cap (X - H)$ is Cg-closed subset of X by **Theorem 4.2** Now $(X - G) \cap (X - H) = X - (G \cup H)$ be Cg-closed set $\Rightarrow G \cup H$ is Cg-open set. Hence union of two Cg-open sets is Cg-open set.

In general finite union of Cg-open sets is Cg-open set.

Theorem 4.4: Intersection of two Cg-open sets is Cg-open set.
Proof: Let G and H be two Cg-open subset of a topological space $(X, \Im)$. Then $X - G$ and $X - H$ be two Cg-closed subsets of X. Hence $(X - G) \cup (X - H)$ be the Cg-closed subset of X by **Theorem 4.1**. Now $(X - G) \cup (X - H) = X - (G \cap H)$ be Cg-closed set $\Rightarrow G \cap H$ is Cg-open set. Hence intersection of two Cg-open sets is Cg-open set.

In general finite intersection of Cg-open sets is Cg-open set.

Remark 4.5: Arbitrary union of Cg-closed sets is may not be Cg-closed set.

Example 4.6: For the set of natural number ℕ, $\mathfrak{I}$ be the collection of all those subset of ℕ whose complement is finite together with the empty set, then $\mathfrak{I}$ is cofinite topology for ℕ. Let $A_n = \{n+1\}$ $\forall$ n ∈ {1, 2, 3, 4…..} be the closed sets, hence Cg-closed subsets in the ℕ. Now let A be the countable union of A_n, i.e. $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup ….$ = {2, 3, 4, 5…} = ℕ − {1} which is not Cg-closed set as A ⊂ ℕ − {1} and ℕ − {1} is C-open set *(as ℕ − {1} is open set in ℕ, and cl(ℕ − {1}) − (ℕ − {1}) = ℕ − (ℕ − {1}) = singleton set {1} which is countable)* but cl(A)= ℕ which is not subset of ℕ − {1}. Hence arbitrary union of Cg-closed sets is may not be Cg-closed set.

Remark 4.7: Arbitrary intersection of Cg-open sets is may not be Cg-open set.

Example 4.8: By **example 4.6**, B_n = ℕ − {n+1} ∀ n ∈ ℕ be the open set, hence Cg-open sets in ℕ. Now let B be the countable intersection of B_n, i. e. $B = B_1 \cap B_2 \cap B_3 \cap B_4$ …..= (ℕ − {2}) ∩ (ℕ − {3}) ∩ (ℕ − {4}) ∩ (ℕ − {5})….. = ℕ – ({2} ∪ {3} ∪ {4} ∪ {5}….) = ℕ − {2, 3, 4, 5…} = {1} which is not a Cg-open set as ℕ − {1} is not Cg-closed set by **Example 4.6**. Hence arbitrary intersection of Cg-open sets is may not be Cg-open set.

Definition 4.9: The intersection of all C-open subsets of a space X containing a set B is called the C-kernel of B and is denoted by C-ker(B).

Lemma 4.10: A subset B of a space X is Cg-closed iff cl(B) ⊂ C-ker(B).
Proof: Let B is a Cg-closed set in X. Then cl(B) ⊂ U whenever B ⊂ U and U is C-open in X. This implies cl(B) ⊂ ∩{U: B ⊂ U and U is C-open in X} i. e. cl(B) ⊂ C-ker(B).
Conversely, let cl(B) ⊂ C-ker(B). This implies cl(B) ⊂ ∩{U: B ⊂ U and U is C-open in X} i. e. cl(B) ⊂ U whenever B ⊂ U and U is C-open in X. This proves that B is Cg-closed.

5. C-NORMAL SPACES

Definition 5.1: A space X is said to be **C-normal** (resp. **normal** [8]) if for every pair of disjoint C-closed (resp. closed) sets J and K in X, there exist disjoint open sets G and H such that J ⊂ G and K ⊂ H.

Remark 5.2: Every normal space is C-normal but not conversely.

Theorem 5.3 : For a topological space X, the following properties are equivalent:
(1) X is C-normal;
(2) for any disjoint J, K ∈ C-C(X), there exist disjoint Cg-open sets G, H such that J ⊂ G and K ⊂ H;
(3) for any J ∈ C-C(X) and any H ∈ C-O(X) containing J, there exists a Cg-open set G of X such that J ⊂ G ⊂ Cg-cl(G) ⊂ H;
(4) for any J ∈ C-C(X) and any H ∈ C-O(X) containing J, there exists an open set G of X such that J ⊂ G ⊂ cl(G) ⊂ H;
(5) for any disjoint J, K ∈ C-C(X), there exist disjoint regular open sets G, H such that J ⊂ G and K ⊂ H.

Proof: (1) ⇒ (2): Since every open set is Cg-open, the proof is obvious.

(2) ⇒ (3): Let $J \in$ C-C(X) and H be any C-open set containing J. Then J, $X - H \in$ C-C(X) and $J \cap (X - H) = \phi$. By (2), there exist Cg-open sets G, F such that $J \subset G$, $X - H \subset F$ and $G \cap F = \phi$. Therefore, we have $J \subset G \subset (X - F) \subset H$. Since G is Cg-open and X – F is Cg-closed, we obtain $J \subset G \subset$ Cg-cl(G) $\subset (X - F) \subset H$.

(3) ⇒ (4): Let $J \in$ C-C(X) and $J \subset H \in$ C-O(X). By (3), there exists a Cg-open set G_0 of X such that $J \subset G_0 \subset$ Cg-cl(G_0) $\subset$ H. Since Cg-cl(G_0) is Cg-closed and $H \in$ C-O(X), cl(Cg-cl(G_0)) $\subset$ H. Put int(G_0) = G, then G is open and $J \subset G \subset$ cl(G) $\subset$ H.

(4) ⇒ (5): Let J, K be disjoint C-closed sets of X. Then $J \subset (X - K) \in$ C-O(X) and by (4) there exists an open set G_0 such that $J \subset G_0 \subset$ cl(G_0) $\subset (X - K)$. Therefore, $H_0 = (X -$ cl(G_0)) is an open set such that $J \subset G_0$, $K \subset H_0$ and $G_0 \cap H_0 = \phi$. Moreover, put G = int(cl(G_0)) and H = int (cl(H_0)), then G, H are regular open sets such that $J \subset G$, $K \subset H$ and $G \cap H = \phi$.

(5) ⇒ (1): This is obvious.

We get a characterization of normal spaces by using Cg-open sets.

Theorem 5.4: For a topological space X, the following properties are equivalent:

(1) X is normal;
(2) for any disjoint closed sets J and K, there exist disjoint Cg-open sets G and H such that $J \subset G$ and $K \subset H$;
(3) for any closed set J and any open set H containing J, there exists a Cg-open set G of X such that $J \subset G \subset$ cl(G) $\subset$ H.

Proof: (1) ⇒ (2): This is obvious since every open set is Cg-open.

(2) ⇒ (3): Let J be a closed set and H be any open set containing J. Then J and (X – H) are disjoint closed sets. There exist disjoint Cg-open sets G and F such that $J \subset G$ and $(X - H) \subset F$. Since X – H is closed, we have $(X - H) \subset$ int(F) and $G \cap$ int(F) = ϕ . Therefore, we obtain cl(G) $\cap$ int(F) = ϕ and hence $J \subset G \subset$ cl(G) $\subset (X -$ int(F)) $\subset$ H.

(3) ⇒ (1): Let J, K be disjoint closed sets of X. Then $J \subset (X - K)$ and (X – K) is open. By (3), there exists a Cg-open set F of X such that $J \subset F \subset$ cl(F) $\subset (X - K)$. Since J is closed, we have $J \subset$ int(F). Put G = int(F) and H = (X – cl(F)). Then G and H are disjoint open sets of X such that $J \subset G$ and $K \subset H$. Hence, X is normal.

Lemma 5.5: A subset G of a space X is Cg-open if and only if $F \subset$ int(G) whenever $F \subset G$ and F is C-closed
Proof: let G be a Cg-open set then X − G is Cg-closed set. Since X − G is Cg-closed iff cl(X − G) $\subset$ X − F whenever $X - G \subset X - F$ and X – F is Cg-open, this implies that X – int(G) $\subset$ X – F whenever $F \subset G$ and F is Cg-closed *(because cl(X – G) = X – int(G))*, i. e. $F \subset$ int(G) whenever $F \subset G$ and F is Cg-closed.

Theorem 5.6: For a space topological X, the following are equivalent:

(1) X is C-normal.
(2) For any disjoint C-closed sets J and K, there exist disjoint g-open sets G and H such that $J \subset G$ and $K \subset H$.

(3) For any disjoint C-closed sets J and K, there exist disjoint Cg-open sets G and H such that $J \subset G$ and $K \subset H$.
(4) For any C-closed set J and any C-open set H containing J, there exists a g-open set G of X such that $J \subset G \subset cl(G) \subset H$.
(5) For any C-closed set J and any C-open set H containing J, there exists a Cg- open set G of X such that $J \subset G \subset cl(G) \subset H$.

Proof: (1) ⇒ (2): Let X be C-normal space. Let J, K be disjoint C-closed sets of X. By assumption, there exist disjoint open sets G, H such that $J \subset G$ and $K \subset H$. Since every open set is g-open, so G and H are g-open sets such that $J \subset G$ and $K \subset H$.

(2) ⇒ (3): Let J and K be two disjoint C-closed sets. By assumption, there exist disjoint g-open sets G and H such that $J \subset G$ and $K \subset H$. Since every g-open set is Cg-open, G and H are Cg-open sets such that $J \subset G$ and $K \subset H$.

(3) ⇒ (4): Let J be any C-closed set and H be any C-open set containing J. By assumption, there exist disjoint Cg-open sets G and H_1 such that $J \subset G$ and $X - H \subset H_1$. By **Lemma 5.5**, we get $X - H \subset int(H_1)$ and $cl(G) \cap int(H_1) = \phi$. Hence $J \subset G \subset cl(G) \subset X - int(H_1) \subset H$.

(4) ⇒ (5): Let J be any C-closed set and H be any C-open set containing J. By assumption, there exist g-open set G of X such that $J \subset G \subset cl(G) \subset H$. Since, every g-open set is Cg-open, there exists Cg-open sets G of X such that $J \subset G \subset cl(G) \subset H$.

(5) ⇒ (1): Let J, K be any two disjoint C-closed sets of X. Then $J \subset X - K$ and $X - K$ is C-open. By assumption, there exists Cg-open set G_1 of X such that $J \subset G_1 \subset cl(G_1) \subset X - K$. Put $G = int(G_1)$, $H = X - cl(G_1)$. Then G and H are disjoint open sets of X such that $J \subset G$ and $K \subset H$.

Theorem 5.6: Let X be a C-normal space. Then a semi-regular subspace Y of X is also C-normal.
Proof: Let X be a C-normal space and Y be a semi-regular subspace of X. Let $J \in C\text{-}C(Y)$ and $H \in C\text{-}O(Y)$ containing J. Since Y is semi regular, so $J \in C\text{-}C(X)$ and $H \in C\text{-}O(X)$. Hence by **Theorem 5.3(4)**, there exists an open set G in X such that $J \subset G \subset cl_X(G) \subset H$. This gives $J \subset (G \cap Y) \subset cl_Y(G \cap Y) \subset H$, where $G \cap Y$ is open in Y and hence Y is C-normal.

6. FUNCTIONS AND C-NORMAL SPACES

Definition 6.1: A function $f : X \to Y$ is said to be:
(1) almost Cg-continuous if for any regular open set U of Y, $f^{-1}(U) \in Cg\text{-}O(X)$;
(2) almost Cg-closed if for any regular closed set J of X, $f(J) \in Cg\text{-}C(Y)$.

Definition 6.2: A function $f : X \to Y$ is said to be:
(1) **C-irresolute (resp. C-continuous [2])** if for any C-open (resp. open) set U of Y, $f^{-1}(U)$ is C-open in X;
(2) **pre-C-closed (resp. C-closed [2])** if for any C-closed (resp. closed) set J of X, f(J) is C-closed in Y.

Theorem 6.3: A function $f : X \to Y$ is an almost Cg-closed surjection iff for each subset P of Y and each regular open set G containing $f^{-1}(P)$, there exists a Cg-open set H such that $P \subset H$ and $f^{-1}(H) \subset G$.

Proof: Necessity. Suppose that f is almost Cg-closed. Let P be a subset of Y and G be a regular open set of X containing $f^{-1}(P)$. Put $H = Y - f(X - G)$, then H is a Cg-open set of Y such that $P \subset H$ and $f^{-1}(H) \subset G$.

Sufficiency: Let J be any regular closed set of X. Then $f^{-1}(Y - f(J)) \subset (X - J)$ and $X - J$ is regular open. There exists a Cg-open set H of Y such that $(Y - f(J)) \subset H$ and $f^{-1}(H) \subset (X - J)$. Therefore, we have $f(J) \supset (Y - H)$ and $J \subset f^{-1}(Y - H)$. Hence, we obtain $f(J) = Y - H$ and f(J) is Cg-closed in Y. Therefore f is almost Cg-closed.

Theorem 6.4: If $f : X \to Y$ is an almost Cg-closed C-irresolute (resp. C-continuous) surjection and X is C-normal, then Y is C-normal (resp. normal).

Proof: Let J and K be any disjoint C-closed (resp. closed) sets of Y. Then $f^{-1}(J)$ and $f^{-1}(K)$ are disjoint C-closed sets of X. Since X is C-normal, there exist disjoint open sets G and H of X such that $f^{-1}(J) \subset G$ and $f^{-1}(K) \subset H$. Put $G_1 = int(cl(G))$ and $H_1 = int(cl(H))$, then G_1 and H_1 are disjoint regular open sets of X such that $f^{-1}(J) \subset G_1$ and $f^{-1}(K) \subset H_1$. By **Theorem 6.3**, there exist Cg-open sets L and M of Y such that $J \subset L$, $K \subset M$. $f^{-1}(L) \subset G_1$ and $f^{-1}(M) \subset H_1$. Since G_1 and H_1 are disjoint, so L and M are also disjoint. It follows from **Theorem 5.3** (resp. **Theorem 5.4**) that Y is C- normal (resp. normal).

Theorem 6.5: If $f : X \to Y$ is a continuous almost Cg-closed surjection and X is a normal space, then Y is normal.

Proof: The proof is similar to that of **Theorem 6.4.**

Theorem 6.6: If $f : X \to Y$ is an almost Cg-continuous pre-C-closed (resp. C-closed) injection and Y is C-normal, then X is C-normal (resp. normal).

Proof: Let J and K be disjoint C-closed (resp. closed) sets of X. Since f is a pre-C-closed (resp. C-closed) injection, f(J) and f(K) are disjoint C-closed sets of Y. Since Y is C-normal, there exist disjoint open sets G and H such that $f(J) \subset G$ and $f(K) \subset H$. Now, put $G_1 = int(cl(G))$ and $H_1 = int(cl(H))$, then G_1 and H_1 are disjoint regular open sets such that $f(J) \subset G_1$ and $f(K) \subset H_1$. Since f is almost Cg-continuous, $f^{-1}(G_1)$ and $f^{-1}(H_1)$ are disjoint Cg-open sets such that $J \subset f^{-1}(G_1)$ and $K \subset f^{-1}(H_1)$. It follows from **Theorem 5.3** (resp. **Theorem 5.4**) that X is C-normal (resp. normal).

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REFERENCES

1. Mesfer H. Alqahtani, F-open and F-closed sets in Topological Spaces, *European Journal of Pure and Applied Mathematics*, vol. 16, No. 2, 2023, 819-832.
2. Mesfer H. Alqahtani, C-open sets on Topological Spaces, arXiv:2305.03166v1 [math .Gn] 4 may 2023.
3. Hamant Kumar, B. S. Sharma and Anuj Kumar, Fg-Closed Sets and F-Normal Spaces, *International Journal of Creative Research Thoughts (IJCRT)*, vol. 12, issue 4, 2024, 577-582.
4. N. Levine, Semi-Open Sets and Semi-Continuity in Topological Spaces, *The American Mathematical Monthly*, vol. 70, No.1, 1963, 36–41.
5. N. Levine, Generalized Closed Sets in Topology, *Rend. Circ. Mat. Palermo*, vol. 19, 1970, 89-96.

6. K. C. Rao and K. Joseph, semi star generalized closed sets, *Bulletin of Pure and Applied Science E*, vol. 19, 2002, 281-290.
7. M. H. Stone, Application of The Theory of Boolean Rings to General Topology, *Trans. Amer. Math. Soc.*, vol. 41, 1937, 375-481.
8. Tietze H., Uber Analysis Situs, *Abh. Math. Semin. Univ. Hambg.*, Vol. 2, 1923, 37-68.