



Generalized Fractional Differential Operators Involving Two \bar{H} -Functions

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Abstract: In this paper we aim to establish certain image formulas of \bar{H} -function under Marichev- Saigo-Maeda fractional differential operators having Appell's function $F_3(\cdot)$ as a kernel. The results are expressed in terms of \bar{H} -function. A number of known and new results can also be obtained as special cases of these results.

Keywords: Marichev-Saigo-Maeda Fractional Differential Operators, \bar{H} -function, Appell function.

I. INTRODUCTION

Fractional calculus is an important branch of mathematical analysis which deals with investigations of integrals and derivatives of arbitrary order. Fractional calculus operators involving the various special functions have been considered by many authors like Miller and Ross [7], Kalla and Saxena [11], Kiryakova [12], McBride [13], Kilbas and Sebastian [14] etc. A Large number of fractional calculus formulas involving H -function and \bar{H} functions have been studied and developed by many authors ([3], [9], [24], [25], [26]).

The Marichev-Saigo-Maeda fractional calculus operators involving Appell function $F_3(\cdot)$ of two variables [22] in the kernel are more general in nature. Generalized fractional differentiation operators of arbitrary order have been introduced by Marichev [4] and later extended and studied by Saigo and Maeda [17], in the following forms;

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{R}$, $x > 0$ and $\text{Re}(\gamma) > 0$ then ,

$$\begin{aligned} (D_{0+}^{\alpha, \alpha', \beta, \beta'; \gamma} f)(x) &= (I_{0+}^{-\alpha', -\alpha, -\beta', -\beta; -\gamma} f)(x) \\ &= \left(\frac{d}{dx}\right)^n (I_{0+}^{-\alpha', -\alpha, -\beta'+n, -\beta, -\gamma+n} f)(x) \quad (\text{Re}(\gamma) > 0; n = [\text{Re}(\gamma) + 1]) \\ &= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx}\right)^n (x^{\alpha'}) \int_0^x (x-t)^{n-\gamma-1} t^{\alpha} \times F_3\left(-\alpha', -\alpha, -\beta'+n, -\beta, n-\gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t) dt \end{aligned} \quad (1)$$

and

$$(D_-^{\alpha, \alpha', \beta, \beta'; \gamma} f)(x) = (I_-^{-\alpha', -\alpha, -\beta', -\beta; -\gamma} f)(x)$$

$$= \left(-\frac{d}{dx}\right)^n \left(I_{-}^{-\alpha', -\alpha, -\beta', -\beta+n, -\gamma+n} f\right)(x) \quad (\operatorname{Re}(\gamma) > 0; n = [\operatorname{Re}(\gamma) + 1])$$

$$= \frac{1}{\Gamma(n-\gamma)} \left(-\frac{d}{dx}\right)^n (x^\alpha) \int_x^\infty (t-x)^{n-\gamma-1} t^{\alpha'} \times F_3 \left(-\alpha', -\alpha, -\beta', n-\beta, n-\gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t) dt$$

(2)

Where $I_{0,+}^{\alpha, \alpha', \beta, \beta', \gamma}$ and $I_{-}^{\alpha, \alpha', \beta, \beta', \gamma}$ are Saigo Maeda Fractional Integral operators [17] defined by,

$$\left(I_{0,+}^{\alpha, \alpha', \beta, \beta', \gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha} \times F_3 \left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t) dt$$

(3)

$$\text{and } \left(I_{0,-}^{\alpha, \alpha', \beta, \beta', \gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha'} \times F_3 \left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t) dt$$

(4)

F_3 the Appell Hypergeometric function [22] of two variables is defined as follows;

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{|m| |n|} \quad (|x| < 1, |y| < 1) \quad (5)$$

where $(\lambda)_n$ is Pochhammer symbol, $(\lambda \in \mathbb{C}; n \in \mathbb{N}_0)$ defined as:

$$(\lambda)_n = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\dots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases}$$

The above series (5) is absolutely convergent for all $x, y \in \mathbb{C}$ with $|x| < 1, |y| < 1$ and for all $x, y \in \mathbb{C} \setminus \{1\}$ with $|x|=1, |y|=1$.

Power function formulas of the above discussed fractional operators are needed for our present study as given in the following lemmas;

Lemma 1: If $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ be such that, $\operatorname{Re}(\gamma) > 0$ and

$\operatorname{Re}(\rho) > \max [0, \operatorname{Re}(-\alpha - \alpha' - \beta' + \gamma), \operatorname{Re}(-\alpha + \beta)]$, then;

$$\left(D_{0,+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1}\right)(x) = \Gamma \left[\begin{matrix} \rho, \rho - \gamma + \alpha + \alpha' + \beta', \rho - \beta + \alpha \\ \rho - \beta, \rho - \gamma + \alpha + \beta', \rho - \gamma + \alpha + \alpha' \end{matrix} \right] x^{\rho + \alpha + \alpha' - \gamma - 1}$$

(6)

$$\text{where, } \Gamma \left[\begin{matrix} a, b, c \\ x, y, z \end{matrix} \right] = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(x)\Gamma(y)\Gamma(z)}$$

Lemma 2: If $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ be such that, $\operatorname{Re}(\gamma) > 0$ and

$\operatorname{Re}(\rho) < 1 + \min [\operatorname{Re}(\beta'), \operatorname{Re}(\gamma - \alpha - \alpha'), \operatorname{Re}(\gamma - \alpha' - \beta)]$, then;

$$\left(D_{-}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1}\right)(x) = \Gamma \left[\begin{matrix} 1 - \rho + \gamma - \alpha - \alpha', 1 - \rho - \alpha' - \beta + \gamma, 1 - \rho + \beta' \\ 1 - \rho, 1 - \rho - \alpha - \alpha' - \beta + \gamma, 1 - \rho - \alpha' + \beta' \end{matrix} \right] x^{\rho + \alpha + \alpha' - \gamma - 1} \quad (7)$$

The \bar{H} -function was introduced by Inayat Hussain [1] and studied by Bushman and Shrivastava [2] is defined and represented in the following manner,

$$\bar{H}_{p,q}^{m,n} [Z] = \bar{H}_{p,q}^{m,n} \left[Z \middle| \begin{matrix} (e_j, E_j; A_j)_{1,n}, (e_j, E_j)_{n+1,p} \\ (f_j, F_j)_{1,m}, (f_j, F_j; B_j)_{m+1,q} \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_L z^\xi \bar{\theta}(\xi) d\xi, \quad (z \neq 0)$$

(8)

where,
$$\bar{\theta}(\xi) = \frac{\prod_{j=1}^m \Gamma(f_j - F_j \xi) \prod_{j=1}^n \{\Gamma(1 - e_j + E_j \xi)\}^{A_j}}{\prod_{j=m+1}^q \{\Gamma(1 - f_j + F_j \xi)\}^{B_j} \prod_{j=n+1}^p \Gamma(e_j - E_j \xi)}$$

(9)

Here L is a contour starting at the point $c - i\infty$ and terminating at the point $c + i\infty$, $\bar{\theta}(\xi)$ contains fractional powers of some of the gamma function. Here z may be real or complex but it is not equal to zero, m, n, p, q are integers such that $1 \leq m \leq q, 1 \leq n \leq p$. Also, $e_j (j=1, \dots, p)$ and $f_j (j=1, \dots, q)$ are complex parameters, $E_j \geq 0 (j=1, \dots, p), F_j \geq 0 (j=1, \dots, q)$, (not all zero simultaneously) and the exponents $A_j (j=1, \dots, n), B_j (j=m+1, \dots, q)$ can take integer values.

Sufficient condition for absolute convergence of the contour integral in (8) and other detail see Inayat Hussain [1] and Buschman and Shrivastava [2].

When the exponents $A_j = B_j = 1, \forall i$ and j , the \bar{H} -function reduces to familiar Fox's H-function defined by Fox [15] and Mathai and Saxena [23].

The series representation of \bar{H} -function [1] is as follows

$$\begin{aligned} \bar{H}_{p',q'}^{m',n'}[z] &= \bar{H}_{p',q'}^{m',n'} \left[z \left| \begin{matrix} (e'_j, E'_j; A'_j)_{1,n'} \\ (f'_j, F'_j)_{1,m'} \end{matrix} \right. \begin{matrix} (e'_j, E'_j)_{n'+1,p'} \\ (f'_j, F'_j; B'_j)_{m'+1,q'} \end{matrix} \right. \\ &= \sum_{g=1}^{m'} \sum_{h=0}^{\infty} \frac{(-1)^h \phi(\eta_{g,h})}{h! F'_g} z^{\eta_{g,h}}, \end{aligned}$$

(10)

where
$$\phi(\eta_{g,h}) = \frac{\prod_{j=1}^{m'} \Gamma(f'_j - F'_j \eta_{g,h}) \prod_{j=1}^{n'} \{\Gamma(1 - e'_j + E'_j \eta_{g,h})\}^{A'_j}}{\prod_{j=m'+1}^{q'} \{\Gamma(1 - f'_j + F'_j \eta_{g,h})\}^{B'_j} \prod_{j=n'+1}^{p'} \Gamma(e'_j - E'_j \eta_{g,h})}$$
 and
$$\eta_{g,h} = \frac{f'_g + h}{F'_g}$$

(11)

II. Main Results

Theorem 1: If $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}, x > 0, T > 0, |\arg z| < \frac{1}{2}\pi T$, such that $\text{Re}(\gamma) > 0$ and $\text{Re}(\rho + \sigma \eta_{g,h} + \delta \xi) > \max [0, \text{Re}(-\alpha - \alpha' - \beta' + \gamma), \text{Re}(-\alpha + \beta)]$, then;

$$\begin{aligned} D_{0,+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\rho-1} \bar{H}_{p',q'}^{m',n'}[\lambda t^\sigma] \bar{H}_{p,Q}^{M,N}[\omega t^\delta] \right) (x) \\ = x^{\rho + \sigma \eta_{g,h} + \alpha + \alpha' - \gamma - 1} \sum_{g=1}^{m'} \sum_{h=0}^{\infty} \frac{(-1)^h \phi(\eta_{g,h})}{h! F'_g} \lambda^{\eta_{g,h}} \end{aligned}$$

$$\times \bar{H}_{P+3, Q+3}^{M, N+3} \left[\omega x^\delta \left| \begin{matrix} (e_j, E_j; A_j)_{1, N}, (e_j, E_j)_{N+1, P} (1-\rho-\sigma\eta_{g,h}, \delta; 1), (1-\rho-\sigma\eta_{g,h}-\alpha-\alpha'-\beta+\gamma, \delta; 1), (1-\rho-\sigma\eta_{g,h}-\alpha+\beta, \delta; 1), \\ (f_j, F_j)_{1, M}, (f_j, F_j; B_j)_{M+1, Q}, (1-\rho-\sigma\eta_{g,h}-\alpha+\gamma-\beta', \delta; 1), (1-\rho-\sigma\eta_{g,h}-\alpha-\alpha'+\gamma, \delta; 1), (1-\rho-\sigma\eta_{g,h}+\beta', \delta; 1) \end{matrix} \right. \right]$$

(12)

Proof: Applying equation (8) and (10) to the left hand side of (12) and then interchanging the order of summation and integration we have,

$$D_{0,+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\rho-1} \bar{H}_{p', q'}^{m', n'} \left[\lambda t^\sigma \right] \bar{H}_{P, Q}^{M, N} \left[\omega t^\delta \right] \right) (x) \\ = \sum_{g=1}^{m'} \sum_{h=0}^{\infty} \frac{(-1)^h \phi(\eta_{g,h})}{h! F_g'} \lambda^{\eta_{g,h}} \times \frac{1}{2\pi i} \int_L \omega^\xi \bar{\theta}(\xi) \left\{ D_{0,+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho+\sigma\eta_{g,h}+\delta\xi-1} \right\} (x) d\xi$$

Now applying the Saigo -Maeda operator (6) we get,

$$= x^{\rho+\sigma\eta_{g,h}+\alpha+\alpha'-\gamma-1} \sum_{g=1}^{m'} \sum_{h=0}^{\infty} \frac{(-1)^h \phi(\eta_{g,h})}{h! F_g'} \lambda^{\eta_{g,h}} \\ \times \frac{1}{2\pi i} \int_L \bar{\theta}(\xi) \left[\frac{\Gamma(\rho+\sigma\eta_{g,h}+\delta\xi)\Gamma(\rho+\sigma\eta_{g,h}+\delta\xi-\gamma+\alpha+\alpha'+\beta')\Gamma(\rho+\sigma\eta_{g,h}+\delta\xi-\beta+\alpha)}{\Gamma(\rho+\sigma\eta_{g,h}+\delta\xi-\gamma+\alpha+\beta')\Gamma(\rho+\sigma\eta_{g,h}+\delta\xi-\gamma+\alpha+\alpha')\Gamma(\rho+\sigma\eta_{g,h}+\delta\xi-\beta)} \right] (\omega x^\delta)^\xi d\xi$$

Rewriting the RHS of above equation in view of definition (8) we arrive at result (12).

Theorem 2: If $\alpha, \alpha', \beta, \beta', \rho \in \mathbb{C}$, $x > 0$, $T > 0$, $|\arg z| < \frac{1}{2}\pi T$, such that $\text{Re}(\gamma) > 0$ and $\text{Re}(\rho + \sigma\eta_{g,h} + \delta\xi) < 1 + \min [\text{Re}(\beta'), \text{Re}(\gamma - \alpha - \alpha'), \text{Re}(\gamma - \alpha' + \beta)]$, then;

$$D_{-}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\rho-1} \bar{H}_{p', q'}^{m', n'} \left[\lambda t^\sigma \right] \bar{H}_{P, Q}^{M, N} \left[\omega t^\delta \right] \right) (x) = x^{\rho+\sigma\eta_{g,h}+\alpha+\alpha'-\gamma-1} \sum_{g=1}^{m'} \sum_{h=0}^{\infty} \frac{(-1)^h \phi(\eta_{g,h})}{h! F_g'} \lambda^{\eta_{g,h}} \\ \times H_{P+3, Q+3}^{M+3, N} \left[\omega x^\delta \left| \begin{matrix} (e_j, E_j; A_j)_{1, N}, (e_j, E_j)_{N+1, P} (1-\rho-\sigma\eta_{g,h}, \delta; 1), (1-\rho-\sigma\eta_{g,h}-\alpha-\alpha'-\beta+\gamma, \delta; 1), (1-\rho-\sigma\eta_{g,h}-\alpha'+\beta', \delta; 1), \\ (f_j, F_j)_{1, M}, (f_j, F_j; B_j)_{M+1, Q}, (1-\rho-\sigma\eta_{g,h}-\alpha'+\gamma-\beta, \delta; 1), (1-\rho-\sigma\eta_{g,h}-\alpha-\alpha'+\gamma, \delta; 1), (1-\rho-\sigma\eta_{g,h}+\beta', \delta; 1) \end{matrix} \right. \right]$$

(13)

Proof: Applying equation (8) and (10) to the left hand side of (13) and then interchanging the order of summation and integration we have,

$$D_{-}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\rho-1} \bar{H}_{p', q'}^{m', n'} \left[\lambda t^\sigma \right] \bar{H}_{P, Q}^{M, N} \left[\omega t^\delta \right] \right) (x) \\ = \sum_{g=1}^{m'} \sum_{h=0}^{\infty} \frac{(-1)^h \phi(\eta_{g,h})}{h! F_g'} \lambda^{\eta_{g,h}} \times \frac{1}{2\pi i} \int_L \omega^\xi \bar{\theta}(\xi) \left\{ D_{0,-}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho+\sigma\eta_{g,h}+\delta\xi-1} \right\} (x) d\xi$$

Now applying the Saigo -Maeda operator (7) we get,

$$= x^{\rho+\sigma\eta_{g,h}+\alpha+\alpha'-\gamma-1} \sum_{g=1}^{m'} \sum_{h=0}^{\infty} \frac{(-1)^h \phi(\eta_{g,h})}{h! F_g'} \lambda^{\eta_{g,h}}$$

$$\times \frac{1}{2\pi i} \int_L \bar{\theta}(\xi) \left[\frac{\Gamma(1-\alpha'-\beta+\gamma-\rho-\sigma\eta_{g,h}-\delta\xi)\Gamma(1+\beta'-\rho-\sigma\eta_{g,h}-\delta\xi)\Gamma(1-\alpha-\alpha'+\gamma-\rho-\sigma\eta_{g,h}-\delta\xi)}{\Gamma(1-\rho-\sigma\eta_{g,h}-\delta\xi)\Gamma(1-\alpha-\alpha'-\beta+\gamma-\rho-\sigma\eta_{g,h}-\delta\xi)\Gamma(1-\alpha'+\beta'-\rho-\sigma\eta_{g,h}-\delta\xi)} \right] (\omega x^\delta)^\xi d\xi$$

Rewriting the RHS of above equation in view of definition (8) we arrive at result (13).

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