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A Review On Different Algebra Norms On Tensor Product Of C*-Algebras

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Abstract: In this review paper, the geometrical properties of algebra norms on the tensor product of C*-algebras are investigated. Initially, algebraic tensor product is discussed, Next, some basic definitions and the structure and properties of different norms on C*-algebras are given. Also, some characterization of nuclear C*-algebras are given and some related properties are discussed.

Keywords: C*-algebra, Tensor Product, C*-norms, Nuclear C*-algebra

I. INTRODUCTION

The theory of C*-algebras is an essential tool in the study of operator *-algebras. It plays a crucial role in the representation theory of a very extensive class of involutive Banach algebras. This theory is also important for the formal calculus with operation rings, the unitary representation theory of groups, a quantum mechanical formalism and abstract ring theory. The modern tensor product theory of Banach spaces was initiated by Grothendieck's fundamental papers [10] and [11] during 1955-56. Grothendieck introduced fourteen 'natural tensor norms' on Banach spaces and Carne in his paper [4] proved that out of these fourteen norms, only four makes the algebraic tensor product a Banach algebra. In 1969, [12] Guichardet discussed about C*-tensor norms and the tensor product of C*-algebras. In 1984, [14] Kaijser and Sinclair studied about the projective tensor product of C*-algebras. In [1] D.P. Blecher mainly worked on the relationship between geometry of operator algebras and their tensor products. In this current study the paper [1] of D.P. Blecher is reviewed and investigated the geometrical properties of algebra norms on the tensor product of C*-algebras. Some basic definitions and the structure and properties of different norms on C*-algebras are discussed and some characterization of nuclear C*-algebras are given and some related properties are discussed. Lastly, some applications of C*norms are mentioned.

For two C*-algebras A and B, the geometrical properties of algebra norms on $A \otimes B$ is discussed. Algebra norm means a norm α which satisfies the property $\alpha\,(u.v) \le \alpha\,(u).\alpha(v)$. Works on tensor products of C*-algebras has concentrated on norms α which makes the completion $A \otimes \alpha B$ into a C*-algebra.

Basic definitions and some results

1. **Algebraic tensor product:**

Let A and B be two C*-algebras. $A \otimes B$ is their algebraic tensor product which is defined as $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ and involution $(\sum_i a_i \otimes b_i)^* = \sum_i a_i^* \otimes b_i^*$.

Normed algebra: An algebra A is said to be a normed algebra if it has a norm that makes it into a normed linear space and the norm also satisfies

a) $\|ab\| \le \|a\|.\|b\|$

b) If A has an identity e then $\|e\| = 1$

Banach algebra: If A is a normed algebra and A is a Banach space (i.e. A, with its norm is complete) then A is called a Banach algebra.

Involution: An involution on an algebra A is a map $x \mapsto x*$ from A to A which satisfies

i) $(x+y)* = x* + y*$
ii) $(\alpha x)* = \bar{\alpha} x*$
iii) $x** = x$
iv) $(xy)* = y*x*$ $for\ all\ x, y \in A\ and\ \alpha \in C$

***-algebra**: An algebra on which an involution has been defined is a *-algebra.
Lemma: when A and B are C*-algebras their algebraic tensor product $A \otimes B$ is a *-algebra.
Adjoint: If $a \epsilon A$, then a^* is called the adjoint of a.
An element a of a *-algebra A is said to be

i) **Self –adjoint or Hermitian if** $a* = a$
ii) **Normal if** $a*a = aa*$
iii) **Unitary if** $a*a = aa* = 1$

C*-algebra: If A is a Banach algebra with involution and also

$$\|aa*\| = \|a\|^2 \quad ...... (1)$$

Then A is called a C*-algebra and the condition (1) is called the "C*-condition".
If for all $a \epsilon A$ we have $\|a*\| = a$ then we say that the involution is isometric.
Representation: Let A be an involutive algebra and H a Hilbert space. A representation of A in H is a map π of A into BL (H) (the algebra of all bounded linear operators on H) such that

$$\pi(x+y) = \pi(x) + \pi(y), \pi(\lambda x) = \lambda\pi(x), \pi(xy) = \pi(x)\pi(y),$$
$$\pi(x*) = \pi(x)*, for\ x, y \in A\ and\ \lambda \in C$$

Faithful representation: A representation is faithful if its kernel is trivial.
Every C*-algebra has a faithful representation. i.e. every C*-algebra is isomorphic to a C*-subalgebra of BL(H) for some Hilbert space H.

Some properties of C*-algebra:

i) In a C*-algebra the involution is isometric.

Proof: using C*-condition and norm algebra for any $a \epsilon A$

$$\|a\|^2 = \|aa*\| \leq \|a\|\|a*\|$$
$$i.e. \|a\| \leq \|a*\| ....(1)$$

For any $a* \epsilon A$, $\|a*\|^2 = \|a*(a*)*\| = \|a*a\| \leq \|a*\|\|a\|$

$$i.e.\ \|a*\| \leq \|a\|..........(2)$$

From (1) and (2) $\|a\| = \|a*\|$. So, involution is isometric.

ii) In a C*-algebra $\|a\| = \sup_{\|x\|\leq 1}\|ax\| = \sup_{\|x\|\leq 1}\|xa\|$

Proof: Clearly, $\sup_{\|x\|\leq 1}\|ax\| \leq \sup_{\|x\|\leq 1}\|a\|\|x\| \leq \|a\|$

And when $x = \frac{a*}{\|a*\|} = \frac{a*}{\|a\|}$ this supremum is attained.

iii) If a C*-algebra A has an identity e, then e=e*

From C*-condition $\|e\| = \|ee*\| = \|e\|^2 \Rightarrow \|e\| = 1$

iv) The C*-condition $\|a*a\| = \|a\|^2$ may be replaced by $\|aa*\| \geq \|a\|^2$

Some Examples of C*-algebra:

1) Let X be a compact space and let C (X) be the Banach space of all complex-valued functions on X with the usual norm $\|f\| = \sup_{x\in X}|f(x)|$. Multiplication in C (X) is defined pointwise: $f.g(x) = f(x)g(x)$ and the involution by complex conjugate $f^*(x) = \overline{f(x)}$. Then C(X) is commutative C*-algebra which has an identity, namely the function e where $e(x) = 1$ for all x ϵ X.
2) Let A be an algebra over the field C of complex numbers. On A = C, the map $z \mapsto \bar{z}$ (where $\bar{z}$ is a complex conjugate of z) is an involution with which A becomes a commutative involutive algebra. Then A is a C*-algebra.

Different norms on the algebraic tensor product of two C*-algebras A and B

a) Projective tensor norm (γ): Given normed spaces A and B, the Projective tensor norm (γ) on $A \otimes B$ is defined by γ (u) = inf $\{\sum_{i=0}^{n}\|a_i\|\|b_i\| : u = \sum_{i=0}^{n} a_i \otimes b_i\}$ where the infimum is taken over all (finite) representations of u.

b) Injective tensor norm (λ): Given normed spaces A and B, the Injective tensor norm λ on $A\otimes B$ is defined by

$$\lambda\left(\sum_{i=0}^{n} a_i \otimes b_i\right) = \sup\left\{\left|\sum_{i=0}^{n} f(a_i)\, g(b_i)\right| : f \in \text{Ball}(A^*),\ g \in \text{Ball}(B^*)\right\}$$

c) Haagerup norm (h):
Given normed spaces A and B, the Haagerup norm on $A\otimes B$ is defined by

$$\|u\|_h = \inf\left\{\left\|\sum_{i=1}^{n} a_i a_i^*\right\|^{1/2} \left\|\sum_{i=1}^{n} b_i^* b_i\right\|^{1/2} : u = \sum_{i=0}^{n} a_i \otimes b_i\right\}$$

d) Reasonable norms: The norms discussed on Grothendieck's fundamental papers [9] and [10] while studying the tensor product theory are called 'reasonable' norms. These are norms α satisfying a certain uniformity condition

$$\alpha\left(S \otimes T(u)\right) \le \|S\|.\alpha(u).\|T\|$$

for all bounded operators S and T between Banach spaces. The injective norm λ and projective norm γ are examples of reasonable norms.

e) Cross norm: Given normed spaces A and B, a norm α on $A \otimes B$ is said to be cross norm if $\alpha(a \otimes b) = \|a\|.\|b\| \ (a \in A, b \in B)$

The Projective tensor norm γ and weak tensor norms on $A\otimes B$ are cross norms. In fact γ is

the largest cross norm on $A \otimes B$; for if α is a cross norm on $A \otimes B$ and $u = \sum_i a_i \otimes b_i$
then $\alpha(u) \le \sum_i a_i \otimes b_i = \sum_i \|a_i\|\|b_i\|$ and so $\alpha(u) \le \gamma(u)$.

f) C*-norms: Let A be a *-algebra which is also a normed algebra. A norm on A that satisfies $\|a * a\| = \|a\|^2$ for all $a \in A$ is called a C*-norm.

If α is an algebra norm defined on an algebra A, we call α a C*-*norm* on A if there is an involution on the α-completion of A making it into a C*- algebra. If A and B are C*-algebras, then there are several norms α that turn $A\otimes_\alpha B$ into a C*-algebra. Two such special norms are:

i) **The least C*-norm $\|\,.\,\|_{min}$:**

Let X and Y be C*-algebras and $f: A \to BL(K_1)$, $g: B \to BL(K_2)$ (where BL (K_1), BL (K_2) are the algebra of all bounded linear operators on K_1 and K_2) be their representations on the Hilbert spaces K_1 and K_2. If $x = \sum_i a_i \otimes b_i \in A \otimes B$ then we define,

$(f \otimes g)(x) = \sum_i f(a_i) \otimes g(b_i) \in BL(K1) \otimes BL(K2) \subset BL(K1 \otimes_\sigma K2)$, where σ is the canonical norm induced by the product of the inner products on K_1 and K_2.

Then $\|\,.\,\|_{min}$ is defined as: $\|x\|_{min} = \|(f\otimes g)(x)\|$ for any $x \in A\otimes B$, where f, g are faithful representations. This definition is independent of the choice of representations.

ii) The greatest C*-norm $\|\,.\,\|_{max}$: This norm was introduced by Guichardet and is defined by $\|x\|_{max} = \sup_\pi \|\pi(x)\|$, where π ranges over the set of all representations of $A\otimes B$ satisfying the condition:

$$\|\pi(a\otimes b)\| \le \|a\|.\|b\| \quad \text{for all } a \in A,\ b \in B.$$

Nuclear C*-algebra: A C*-algebra A is called nuclear if for every C*-algebra B, $\|\,.\,\|_{\mathbf{min}} = \|\,.\,\|_{\mathbf{max}}$ on A ⊗ B. Finite dimensional C*-algebras are nuclear. In particular, $\mathcal{M}_n$ the C*-algebra of all complex n×n matrices are nuclear.

The norms are ordered as follows

$$\lambda \le \|\,.\,\|_{\mathbf{min}} = \|\,.\,\|_{\mathbf{max}} \le \|\,.\,\|_h \le \gamma$$

Positive elements: Let A be a C*-algebra. An element $a \in A$ is said to be positive if it is hermitian and satisfies the conditions

i) $sp(a) \subseteq R^+$

ii) a is of the form bb^* for some $b \in A$

iii) a is of the form h^2 for some hermitian $h \in A$.

The set of positive elements of A is denoted by A^+.

Positive map: If τ is a subset of a C*-algebra A, τ^+ denote the set of all positive elements in A which are also in τ. The linear map $\tau : A \to B$ of C*-algebras is said to be positive if $\tau(A^+) \subseteq B^+$.

Completely positive map:

If the maps $T_n : \mathcal{M}_n(A) \to \mathcal{M}_n(B) : [a_{ij}] \mapsto [Ta_{ij}]$ are all positive where $\mathcal{M}_n(A)$, $\mathcal{M}_n(B)$ are C*-algebras of all complex n×n matrices then T is said to be completely positive.

If A or B is commutative and T is positive then T is completely positive.

The set of all self-adjoint elements in a *-algebra B will be denoted $B_{s.a.}$.

Lemma: If A and B are two C*-algebras then $(A\otimes B)_{s.a.} = A_{s.a.} \otimes B_{s.a.}$ as real spaces.

Proof: Let u be a self-adjoint element in $(A\otimes B)$. i.e. $u \in (A\otimes B)_{s.a.}$.

Let u = $a_i \otimes b_i$ where $a_i \in A$, $b_i \in B$

Then u = u* where u* = $a_i^* \otimes b_i^*$

$\Rightarrow a_i \otimes b_i = a_i^* \otimes b_i^*$

$\Rightarrow a_i = a_i^*,\ b_i = b_i^*$

$\Rightarrow a_i \in A_{s.a},\ b_i \in B_{s.a}$

So $a_i \otimes b_i \in A_{s.a} \otimes B_{s.a} \Rightarrow u \in A_{s.a.} \otimes B_{s.a.}$

Therefore, $(A\otimes B)_{s.a.} \subseteq A_{s.a.} \otimes B_{s.a.}$ ……(1)

Conversely, let $a \in A_{s.a}$, and b $\in B_{s.a}$

So $a = a^*$ and $b = b^*$

Therefore, $a \otimes b \in A_{s.a.} \otimes B_{s.a.}$.

$(a \otimes b)* = a* \otimes b* = a \otimes b$

$\Rightarrow a \otimes b$ is a self-adjoint element of $(A\otimes B)$

$\Rightarrow a\otimes b \in (A\otimes B)_{s.a}$

So $A_{s.a} \otimes B_{s.a} \subseteq (A\otimes B)_{s.a.}$ ....(2)

From (1) and (2) we have $(A\otimes B)_{s.a.} = A_{s.a} \otimes B_{s.a}$

A linear map T_i is defined as T_i: $A_i \to B_i$ (i = 1, 2) where A_i, B_i are C*-algebras.

Then, $T_1 \otimes T_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$ such that $a_1 \otimes a_2 \mapsto T_1 a_1 \otimes T_2 a_2$ $for\ a_1 \in A_1\ and\ a_2 \in A_2$

Completely positive uniform tensor norm:

A tensor norm α is said to be completely positive uniform if whenever

T_i: $A_i \to B_i$ ((i = 1, 2) where A_i, B_i are C*-algebras) are completely positive linear maps of C*-algebras, then $T_1 \otimes T_2$ has an extension

$T_1 \otimes_\alpha T_2$: $A_1 \otimes_\alpha A_2 \to B_1 \otimes_\alpha B_2$ satisfying $\| T_1 \otimes_\alpha T_2 \| \le \|T_1\|.\|T_2\|$

λ, $\|\cdot\|_{min}$, $\|\cdot\|_{max}$, $\|\cdot\|_h$, γ these are completely positive uniform.

Characterization of Nuclear C*-algebra

Nuclear C*-algebra:

A nuclear C*-algebra is a C* algebra A such that the injective and projective C*-cross norms on $(A\otimes B)$ are the same for every C* algebra B. For example, abelian C* algebras are nuclear.

Approximation property:

Let X be a Banach space. Then X is said to have an approximation property if for every compact set K ⊂ X and every ε > 0, there is an operator T: X → X of finite rank so that $\|Tx - x\| \le \varepsilon$ for every x ϵ K.

Bounded approximation property: Let X be a Banach space and let $1 \le \lambda < \infty$. We say that X has the bounded approximation property if for every compact set K ⊂ X and every ε > 0, there is an operator $T: X \to X$ of finite rank so that $\|Tx - x\| \le \varepsilon$ for every x ϵ K and $\|T\| \le \lambda$

Von Neumann algebra: Von Neumann algebra is a subalgebra A of the algebra B (H) of bounded linear operators on a complex Hilbert space H, such that the adjoint operator of any operator in A is also in A, and A is closed in the strong operator topology in B (H).

Enveloping Von Neumann algebra: In operator algebras, the enveloping Von Neumann algebra of C*-algebra is a Von Neumann algebra that contains all the operator-algebric information.

Banach A-bimodule: Let A be an algebra over F, and M a linear space over F. M is said to be a left A-module if a mapping $(a, m) \to am$ of $A \times M$ into M is specified which satisfies the following axioms:

a) For each fixed $a \in$ A, the mapping $m \to am$ is linear on M;
b) For each fixed $m \in M$, the mapping $a \to am$ is linear on A;
c) $a_1(a_2m) = (a_1a_2)m$ $(a_1, a_2 \in A, m \in M)$

The specified mapping $(a, m) \to am$ of $A \times M$ is called the module multiplication.

Similarly, M is said to be a right A-module if a mapping $(a, m) \to ma$ of $A \times M$ into M is specified which satisfies the following axioms:

a) For each fixed $a \in A$, the mapping $m \to ma$ is linear on M;
b) For each fixed $m \in M$, the mapping $a \to ma$ is linear on A;

c) $(m\, a_1)a_2) = m(a_1a_2)$ $(a_1, a_2 \epsilon A, m \epsilon M)$

M is said to be an A-bimodule if it is both a left A-module and a right A-module and the module multiplication is related by the axiom:

$$a(mb) = (am)b \ (a, b \epsilon A, m \epsilon M)$$

If M be a normed linear space over F then M is said to be a normed left A-module if M is a left A-module and there exist a constant K such that

$$\|am\| \leq K \|a\| \|m\| \ (a \epsilon A, m \epsilon M)$$

Similarly, we can define a normed right A-module and a normed A-bimodule is an A-bimodule which is both a normed left A-module and a normed right A-module.

A normed left A-module is called a Banach left A-module if it is complete as a normed linear space. Similarly, for Banach right A-modules and Banach A-bimodules.

Characterization of nuclear C*-algebra

i) A C*-algebra is A Nuclear if and only it has the completely positive approximation property i.e. the identity operator in A can be approximated in the strong operator topology by linear operators of finite rank with norm not exceeding 1, and with the additional property of 'complete positivity'.
ii) A C*-algebra is Nuclear if and only if its enveloping Von Neumann algebra is injective.
iii) Every nuclear C*-algebra has the approximation and bounded approximation properties.
iv) Let A be a C*-algebra. Then A has the extension property for v if the norm v on $A \otimes B$ extends to $C \otimes B$ for every C*-algebra B and for every C*-algebra C which contains A. A nuclear algebra has the extension property * for v.
v) A C*-algebra is Nuclear if and only if it is amenable as a Banach algebra.

Proposition: A C*-algebra A is Nuclear if and only if for all C*-algebras B, and all completely positive uniform tensor norms α, the canonical map ε_α: $A \otimes_\alpha B \to A \otimes_\lambda B$ is injective.

Proof: Suppose that for all C*-algebras B, and all completely positive uniform tensor norms α, the canonical map ε_α: $A \otimes_\alpha B \to A \otimes_\lambda B$ is injective. Let $\alpha = \| \cdot \|_{max}$ .Then the canonical surjection ε_α: $A \otimes_{max} B \to A \otimes_{min} B$ is one-to-one [as the elements of both the domain and co-domain are same, only norms are different] and consequently an isometry. i.e. $\| \cdot \|_{min} = \| \cdot \|_{max}$. Hence A is nuclear.

Conversely, suppose that A is nuclear. Then by the characterization i) of nuclear C*-algebra it is equivalent to the existence of a net (T_v) of completely positive contractive finite rank operators on A converging strongly to the identity mapping I_A. Let B be a C*-algebra and α be a completely positive uniform tensor norm. Suppose u ϵ ker ε_α where ε_α: $A \otimes_\alpha B \to A \otimes_\lambda B$ is a canonical map. Let $f \epsilon (A \otimes_\alpha B)^*$ and for each v, putting $f_V = f \circ (T_v \otimes I_B)$.

$$\|f_v\| = \| f \circ (T_v \otimes I_B).\| \leq \|f\|. \|(T_v \otimes I_B).\| = \|f\|.\|T_v\|.\|I_B\| = \|f\|.\|T_v\|$$

So the net (f_V) is uniformly bounded and so $f_V \to f$ in the weak* topology. Thus $f(u) = 0$ since $f_V(u) = 0$ for each v; since f was chosen arbitrarily, we see that u = 0. So ker $\varepsilon_\alpha = 0$. Hence ε_α is injective.

Proposition: The Haagerup norm is a completely positive uniform algebra tensor norm, and the map ε_h: $A \otimes_h B \to A \otimes_\lambda B$ is always injective.

Approximate identity: Let A be a normed algebra over the field F. A left approximate identity for A is a net $\{e(\lambda)\}_{\lambda \epsilon \Lambda}$ in A such that $e(\lambda)x \to x \ (x \epsilon A)$

A net $\{e(\lambda)\}_{\lambda \epsilon \Lambda}$ in A is bounded if there exist a positive constant k such that

$$\| e(\lambda)\| \leq k \ (\lambda \epsilon \Lambda).$$

A right approximate identity for A is a net $\{e(\lambda)\}_{\lambda \epsilon}$ in A such that $xe(\lambda) \to x \ (x \epsilon A)$

A two-sided approximate identity is a net which is both a left as well as right approximate identity.

Definition: Let B be a C*-algebra. α and α′ are norms on B. If $\alpha'(b) \geq \alpha(b)$ for each $b \epsilon B$ then we say that α' dominates α and write as $\alpha' \geq \alpha$.

***-homomorphism:** Let $(A, *)$ and $(B, \times)$ be *-algebras. A *-homomorphism ϕ is a homomorphism ϕ of A into B such that $\phi(a^*) = (\phi(a))^\times$ $(a \epsilon A)$

If T: $B_1 \to B_2$ is a contraction between unital Banach algebras B_1 and B_2 which is unital i.e. it preserves the identity then $T(H(B_1)) \subset H(B_2)$

Lemma: Let A be a C*algebra and B a Banach algebra, and let T: $A \to B$ be a contraction with dense range, mapping some two-sided approximate identity for A to a two-sided approximate identity for B. Then there exist an involution on B such that B is a C*-algebra and T is involution preserving.

Corollary 1: Let A be a C*-algebra. B a Banach algebra and suppose θ: $A \to B$ is a contractive homomorphism. Then θ (A) possesses an involution which makes it into a C*-algebra isometrically *-isomorphic to A/ker θ, and θ is a *-homomorphism onto θ (A)

Corollary 2: Let A be an algebra. Then any algebra norm on A dominated by a C*-norm is itself a C*-norm, and the canonical contraction between the completions is surjective and involution preserving.

Theorem: Let α be a completely positive uniform algebra tensor norm, which is a C*-norm on $A_0 \otimes B_0$ for some pair of C*-algebras A_0 and B_0 (where neither A_0 nor B_0 equals C). Then α is a C*-tensor norm.

Proof: Let A_0, B_0 be two C*-algebras (where neither A_0 nor B_0 equals C). Let ϕ and ϕ' be two different states on A_0 and ψ and ψ' be two different states on B_0. Define two positive contractions

$$J_1: A_0 \to l_2^\infty : a \mapsto (\phi(a), \phi'(a))$$

$$J_2: B_0 \to l_2^\infty : b \mapsto (\psi(b), \psi'(b))$$

Since J_1, J_2 have commutative ranges they are completely positive and surjective. The uniformity implies that the map

$$J_1 \otimes_\alpha J_2: A_0 \otimes_\alpha B_0 \to l_2^\infty \otimes_\alpha l_2^\infty$$

is contractive and surjective. Also, it preserves a two-sided approximate identity, and so the lemma implies that $l_2^\infty \otimes_\alpha l_2^\infty$ is a C*-algebra with usual involution.

Now let A and B be two unital C*-algebras, and let us choose elements $a \in (\text{Ball } A)_+$ and $b \in (\text{Ball } B)_+$.Consider the positive unital contractions

$$\chi_a : l_2^\infty \to A: (\xi_1, \xi_2) \mapsto \xi_1.a + \xi_2.(1 - a)$$

and $\chi_b : l_2^\infty \to B: (\xi_1, \xi_2) \mapsto \xi_1.b + \xi_2.(1 - b)$

Since l_2^∞ is commutative these maps are completly positive and so $\chi_a \otimes_\alpha \chi_b$ is a unital contraction by the uniformity.

Thus $\chi_a \otimes_\alpha \chi_b ((1, 0) \otimes (1, 0)) = a \otimes b$ is hermitian and we conclude that $A_{s.a} \otimes_R B_{s.a} \subset H(A \otimes_\alpha B)$

Now, $(A_{s.a} \otimes_R B_{s.a}) + i.(A_{s.a} \otimes_R B_{s.a})$ is dense in $A \otimes_\alpha B$, and so

$$H(A \otimes_\alpha B) + i\, H(A \otimes_\alpha B) = A \otimes_\alpha B$$

Hence $(A \otimes_\alpha B)$ is a V-algebra. Hence by Vidav Palmer theorem $A \otimes_\alpha B$ is a C*-algebra with usual involution.

Suppose A and B are arbitrary C*-algebras and let (e_υ) and (f_μ) be positive two-sided approximate identities in A and B respectively. Let $A^1 = A$ if A has a unit, otherwise let A^1 be the C*-algebra obtained by adjoining an identity to A. Similarly, we define B^1.

Define an algebra norm on $A^1 \otimes B^1$ by

$$\tilde{\alpha}(u) = \sup\{\alpha(u.v): v \in A \otimes B, \alpha(v) \le 1\}$$
$$= \lim(\upsilon, \mu)\, \alpha((e\upsilon \otimes f\mu).u.(e\upsilon \otimes f\mu)).$$
$$\text{That } \tilde{\alpha}(u) = 0 \Rightarrow u = 0$$

If α is a C*-norm on $A \otimes B$, then $\tilde{\alpha}$ is the unique C*-norm on $A^1 \otimes B^1$ extending α.

By the uniformity $\tilde{\alpha} \le \alpha$ on $A^1 \otimes B^1$ and by corollary 2 $\tilde{\alpha}$ is a C*-norm. Since $A \otimes_\alpha B$ is embedded isometrically in $A^1 \otimes_{\tilde{\alpha}} B^1$ we find that α is a C*-norm on $A \otimes B$; because A and B were arbitrary, α is a C*-tensor norm.

Some applications of C*-norms in real life world are:

1. The C*norm ensures stability and continuity of measurements. In quantum computing, they help formalize quantum states and operations, especially in quantum error correction and quantum information theory.

2. In some cases of signal processing, *C*-algebras and their norms help describe systems of filters and transformations. The C**-norm ensures that these transformations behave predictably under composition and limits crucial for designing stable systems.
3. In engineering, particularly in robust control, operator algebras with C*-norms are used to model and analyse systems with uncertainties. The norm provides a way to measure the "size" of perturbations and ensure system stability.

4. This is a modern approach to geometry where spaces are described by noncommutative algebras. The C*-norm is vital in defining the "shape" and "distance" in these abstract spaces, with applications in string theory and the standard model of particle physics.

Conclusion: In this review paper we have discussed about the geometrical properties of C*-algebras, also some structures and properties of different norms, characterization of nuclear C*-algebras and related properties. Lastly, some applications of C*-norm are mentioned.

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