



INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

Distal Points For Borel Measures

¹SAKINA RIZVI, ²Dr. CHITRA SINGH

¹(Research Scholar), ¹Department of Mathematics,

¹Rabindranath Tagore University, Bhopal (Madhya Pradesh) India.

²(Supervisor), ²Department of Mathematics

²Rabindranath Tagore University, Bhopal (Madhya Pradesh) India.

Abstract: We introduce the notion of distal point for a Borel measure with respect to a given Homeomorphism. We Prove that a point is distal for every non atomic Borel probability measure if and only if it is countably-distal for the given homeomorphism. We prove that the distal point set of a measure is a Borel set. We study the distal measures (i.e. measures for which every point is distal) and prove that they are approximated with respect to the weak* topology by ones with invariant support. Furthermore, the distal measures are dense in the space of measures just when the ones with full support are. Afterwards, we consider the almost distal measures (i.e. measures for which almost every point is distal) and exhibit one which is not distal. Moreover, a circle homeomorphism has an almost distal measure with full support if and only if it is distal. In particular, every countably distal circle homeomorphism is distal. We prove for circle homeomorphisms with rational rotation number that the almost distal measures are precisely the distal ones. Finally, we prove that every homeomorphism with distal measures has uncountably many almost periodic points and those with almost distal measures on compact spaces have infinitely many nonwandering points.

Keywords: Distal Homeomorphism, Distal Measure, almost distal measure.

Introduction:

Let $f: X \rightarrow X$ be a homeomorphism of a metric space X . A point $x \in X$ is distal if

$$\inf d(f^n(x), f^n(y)) > 0 \quad (n \in \mathbb{Z})$$

for every $y \in X \setminus \{x\}$. Equivalently, if $P(x) = \{x\}$ where

$$P(x) = \{y \in X: \inf d(f^n(x), f^n(y)) = 0\} \quad (1)$$

is the proximal cell of x (occasionally we write $P_f(x)$ to indicate the dependence on f). Consequently, a necessary condition for x to be distal is that $\mu(P(x)) = 0$ for every Borel probability measure μ which is nonatomic (i.e. without points of positive mass).

It is natural to ask if the latter condition is also sufficient for a point x to be distal. Nevertheless, the answer is negative by the following counter example.

Example:

There is a compact metric space X , a homeomorphism $f: X \rightarrow X$ and a point $x \in X$ which is not distal for f though $\mu(P(x)) = 0$ for every nonatomic Borel probability measure μ of X .

Proof:

Denote by $A = \{z = (\theta, r) \in \mathbb{C} : 1 \leq |z| \leq 2\}$ the closed annulus of $\mathbb{C}$. Define $F: A \rightarrow A$ by $F(\theta, r) = (\theta + \alpha \pmod{1}, (r-1)^2 + 1)$ where α is an irrational number.

Clearly F is a homeomorphism of A . Define $X = \partial A \cup \{p_n : n \in \mathbb{Z}\}$ where ∂A denotes the boundary of A and $p_n = F^n(0, \frac{1}{3})$ for $n \in \mathbb{Z}$. Hence X is a compact invariant set of F . Now define the homeomorphism $f : X \rightarrow X$ by $f = F|_X$. Then, $x = (0, 0)$ is not distal since $P(x) = \{(0, 0), (0, \frac{1}{3})\}$ but $\mu(P(x)) = 0$ for every nonatomic Borel probability measure μ is since $P(x)$ has two elements.

However, we will show that the condition $\mu(P(x)) = 0$ for every nonatomic measure μ still implies some distality for x . For this we introduce the following definition closely related to the n -expansive and countable expansive homeomorphisms [5,7,9].

Definition:

Let $f : X \rightarrow X$ be a homeomorphism of a metric space X . We say that $x \in X$ is a countable – distal point of f if $P(x)$ is a countable subset of X . We say that f is countable – distal if every point x is.

Theorem: 1

Let $f : X \rightarrow X$ be a homeomorphism of a Polish metric space X . We say that $x \in X$ is a countable- distal point of f if and only if $\mu(P(x)) = 0$ for every nonatomic Borel probability measure μ of X .

Proof:

It is clear that every countable-distal point x satisfies $\mu(P(x)) = 0$ for every nonatomic Borel probability measure μ . Conversely, suppose that $\mu(P(x)) = 0$ for every nonatomic Borel probability measure μ but $P(x)$ is uncountable. We have that $P(x)$ is a Borel set by Corollary : For every homeomorphism $f : X \rightarrow X$ of a metric space X and every $x \in X$ one has that $P(x)$ is an $F_{\sigma\delta}$ subset of X . In particular, $P(x)$ is a Borel set.

Then, by Theorem 2.8 on p. 12 of [11], there is a Cantor set $C \subseteq P(x)$. On the other hand, by Theorem 8.1 on p. 53 of [11], there is a nonatomic Borel probability μ with $\text{supp}(\mu) \subseteq C$. However, μ is nonatomic so $\mu(P(x)) = 0$ thus $\mu(C) = 0$ which is absurd. This contradiction ends the proof.

Definition:

Let μ be a Borel measure of metric space X . A distal point μ with respect to a homeomorphism $f : X \rightarrow X$ is a point $x \in X$ such that $\mu(P(x)) = 0$. The distal point set of μ with respect to f is the set formed by the distal points of μ with respect to f .

Theorem: 2

For every homeomorphism of a compact metric space, the distal point set of every Borel probability measure is a Borel set.

Proof:

By Corollary (For every homeomorphism $f : X \rightarrow X$ of a metric space X and every $x \in X$ one has that $P(x)$ is an $F_{\sigma\delta}$ subset of X . In particular, $P(x)$ is a Borel set) we have that the map $\phi : X \rightarrow \mathbb{R}$ is defined by $\phi(x) = \mu(P(x))$ for $x \in X$ is well-defined.

On the other hand, the set of distal points of μ is precisely $\phi^{-1}(0)$ and $\phi_{n-1} \xrightarrow{p.p} \uparrow \phi$ by Lemma : For every $x \in X$ and $\epsilon > 0$ one has:

1. $P_i[x, \epsilon]$ is closed;
2. $P[x, \epsilon] = \bigcup_{i \in \mathbb{N}} P_i[x, \epsilon]$
3. $P(x) = \bigcap_{n \in \mathbb{N}^+} P[x, n^{-1}]$

Since each ϕ_{n-1} is Borel by Lemma (If $f : X \rightarrow X$ is a homeomorphism of a compact metric space X , then ϕ_ϵ is a Borel map for every $\epsilon > 0$) we obtain that ϕ is Borel too. Hence $\phi^{-1}(0)$ is a Borel set.

Definition:

A distal measure of a homeomorphism of metric space X is a Borel probability measure whose distal point set is X .

A Borel probability measure μ of a metric space X is invariant respect to a continuous map $f : X \rightarrow X$ if $\mu(f^{-1}(B)) = \mu(B)$ for every Borel set B . The invariance of the measure is not required in the previous definition. On the other hand, not every probability measure is distal, a necessary condition for a measure to be distal is that the measure be nonatomic. This condition is also sufficient for countable – distal homeomorphism.

On the other the distal measures are dense in $M(X)$ for countable – distal homeomorphisms on complete separable metric spaces without isolated points. This follows because the distal measures are precisely the nonatomic ones which in turn are dense in $M(X)$ by corollary 8.1 in [11]. Although every space X exhibiting a homeomorphism $f : X \rightarrow X$ whose distal measure are dense in $M(X)$ has no isolated points, we do not know if such homeomorphism f is countable – distal. However, it is possible to obtain some information of the set of distal measures of a given homeomorphism.

The support of a Borel probability μ is the set $\text{supp}(\mu)$ of points $x \in X$ such that $\mu(U) > 0$ for any neighborhood U of x . It follows that $\text{supp}(\mu)$ is a non empty compact subset of X . A Borel Probability measure μ of X has full or invariant support if $\text{supp}(\mu) = X$ or $f(\text{supp}(\mu)) = \text{supp}(\mu)$ respectively. An invariant measures has invariant support but not conversely, Also we not known if every homeomorphism with distal measures has necessarily an invariant distal measure. A partial answer is stated in the theorem below.

Theorem: 3

Every distal measure of homeomorphism of a compact metric space X can be approximated by ones with invariant support. In addition, the distal measures are dense in $M(X)$ if and only if the ones with full support are.

Proof :

A topological space Y is a Baire space if the intersection of each countable family of open and dense subsets in Y is dense in Y . We say that $A \subseteq Y$ is a Baire subset if A is a Baire space with respect to the topology induced by Y . Hereafter we denote by $M_{\text{dis}}(f)$ the set of distal measures of f .

Let $f: X \rightarrow X$ be a homeomorphism with distal measures of a compact metric space X . By Lemma (For every homeomorphism with distal measures $f : X \rightarrow X$ of a compact metric space X there is a dense subset R of $M_{\text{dis}}(f)$ such that $\text{supp}(\mu) = D(f)$ for all $\mu \in R$.) there is a dense subset of the space of distal measures all of whose elements have support equal to $D(f)$. Since $D(f)$ is invariant by Corollary (The measure distal center is a compact invariant set.), This Completes the proof

Now suppose that $M_{\text{dis}}(f)$ is dense in $M(X)$. Then, $D(f) = X$ (by Lemma5 in [8]) so, by Lemma (For every homeomorphism with distal measures $f : X \rightarrow X$ of a compact metric space X there is a dense subset R of $M_{\text{dis}}(f)$ such that $\text{supp}(\mu) = D(f)$ for all $\mu \in R$.) there is a dense subset R of $M_{\text{dis}}(f)$ such that $\text{supp}(\mu) = X$ for all $\mu \in R$. Since $M_{\text{dis}}(f)$ is dense in $M(X)$, we have that R is dense in $M(X)$ and we are done.

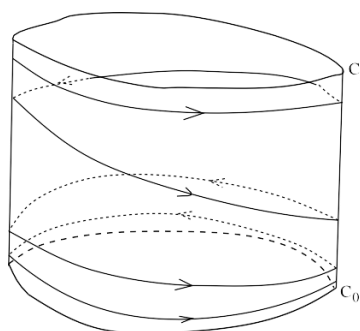


Fig 1: An almost distal measure which is not distal.

Definition:

An almost distal measure of a homeomorphism of a metric space X is a Borel probability measure whose distal point set has a full measure.

Example:

There is a homeomorphism of the two-torus T^2 exhibiting an almost distal measure which is not distal.

Proof:

We obtain T^2 by identifying the circle C_1 and C_2 in Fig. 1. Denote by C the circle in T^2 resulting by such an identification. Consider the flow in T^2 as in that figure. The circle C is the set of equilibrium points of this flow, with the remainder orbits spiraling forward and backward to C as described in the figure. The homeomorphism is precisely the time-1 map of this flow.

If suitable chosen, the flow has the property that this time-1 map satisfies $P(x) \cap C = \{x\}$ (for $x \in C$) and $C \subseteq P(x)$ (for $x \in T^2 \setminus C$). Let μ be the Lebesgue measure of the circle C . It follows that $\mu(P(x)) = 0$ (for $x \in C$) and $\mu(P(x)) = 1$ (for $x \in T^2 \setminus C$). From this we get that μ is almost distal. Since C is a proper subset of T^2 , μ is not distal.

Theorem: 4

A circle homeomorphism has an almost distal measure with full support if and only if it is distal. In particular, every countably-distal circle homeomorphism is distal. Every almost distal measure of a circle homeomorphism with rational rotation number is distal.

Proof:

To finish we will give two dynamical consequences of the existence of distal or almost distal measures. First recall that a stable class of a homeomorphism $f : X \rightarrow X$ is a subset equals to $\{y \in X : \lim_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0\}$ for some $x \in X$. On the other hand, a Borel measure μ of X is positively expansive if there is $\delta > 0$ such that $\mu(\{y \in X : d(f^n(x), f^n(y)) \leq \delta \text{ for all } n \in \mathbb{N}\}) = 0$ for every $x \in X$.

Every continuous map with positively expansive invariant measures has uncountably many stable classes [1]. Below we will obtain the same conclusion for homeomorphisms with almost distal measures (invariant or not).

Theorem: 5

Every homeomorphism with almost distal measures of a metric space has uncountably many stable classes.

Proof: Let $f : X \rightarrow X$ be a homeomorphism of a compact metric space X exhibiting an almost distal measure μ . Suppose by contradiction that the set of stable classes of f is countable. It is apparent that the stable classes form a partition of X . Moreover, such classes are Borel sets (see the proof of Lemma 2.4 in [1]). It follows that $\mu(Ws(x)) > 0$ for some $x \in X$ where $Ws(x) = \{y \in X : \lim_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0\}$ for every $x \in X$. Since μ is almost distal, there is $y \in Ws(x)$ such that y is a distal point of μ . Now fix $x^- \in Ws(x)$. Then, $d(f^n(x^-), f^n(x)) \rightarrow 0$ and, since $y \in Ws(x)$, we also have $d(f^n(x), f^n(y)) \rightarrow 0$ as $n \rightarrow \infty$. From this we obtain $d(f^n(x^-), f^n(y)) \rightarrow 0$ as $n \rightarrow \infty$. We conclude that $x^- \in Ws(y)$ and so $Ws(x) \subseteq Ws(y)$. But clearly $Ws(y) \subseteq P(y)$ so $Ws(x) \subseteq P(y)$. As y is a distal point of μ , we get $\mu(P(y)) = 0$ and so $\mu(Ws(x)) = 0$ which is absurd. Therefore, f has uncountably many stable

For the second consequence, we recall that a subset $F \subset \mathbb{Z}$ is syndetic if there is $l > 0$ such that $[n, n + l] \cap F \neq \emptyset$ for every $n \in \mathbb{Z}$. We say that $x \in X$ is an almost periodic point of a homeomorphism $f : X \rightarrow X$ if for every neighborhood U of x there is a $F \subset \mathbb{Z}$ syndetic such that $f^n(x) \in U$ for every $n \in F$.

We say that x is a non wandering point of f if $\bigcup_{n \in \mathbb{N}^+} f^n(U) \neq \emptyset$ for every neighborhood U of x . The set of almost periodic (resp. non wandering) points is denoted by $AP(f)$ (resp. $\Omega(f)$). The latter is often referred to as the non wandering set of f . Every almost periodic point is non wandering.

Every distal homeomorphism of a compact metric space satisfies that every point is almost periodic (and so non wandering too) [3]. In light of this result, it is tempting to say that every homeomorphism with distal or almost distal has many almost periodic or non-wandering points. This will be obtained in our last result.

Theorem: 6

Every homeomorphism with distal measures of a metric space has uncountably many almost periodic points. Every homeomorphism with almost distal measures of a compact metric space has infinitely many non wandering points.

Some important lemmas

We start with some basic notation. Let $f: X \rightarrow X$ be a homeomorphism of a metric space X . For all $x \in X$ and $\epsilon > 0$ then $P[x, \epsilon]$

$$= \{y \in X : d(f^i(x), f^i(y)) \leq \epsilon \text{ for some } i \in \mathbb{Z}\}$$

$$P_i[x, \epsilon] = \{y \in X : d(f^i(x), f^j(y)) \leq \epsilon \text{ for some } -I \leq j \leq i\}, \forall i \in \mathbb{N}.$$

Lemma 1. For every $x \in X$ and $\epsilon > 0$ one has:

1. $P_i[x, \epsilon]$ is closed;
2. $P[x, \epsilon] = \bigcup_{i \in \mathbb{N}} P_i[x, \epsilon]$;
3. $P(x) = \bigcap_{n \in \mathbb{N}^+} P[x, n^{-1}]$

Proof:

To prove 1 take a sequence $x_k \in P_i[x, \epsilon]$ with $x_k \rightarrow z$ for some $z \in X$. Since $x_k \in P_i[x, \epsilon]$, there exists sequence $-i \leq j_k \leq i$ satisfying $d(f^{j_k}(x), f^{j_k}(x_k)) \leq \epsilon, \forall k \in \mathbb{N}$.

assume that $j_k = j$ for some fixed $-i \leq j \leq i$ yielding $d(f^j(x), f^j(y)) \leq \epsilon, \forall k \in \mathbb{N}$

Letting $k \rightarrow \infty$ above obtain $d(f^j(x), f^j(y)) \leq \epsilon$ for some $-i \leq j \leq i$, this proves $z \in P_i[x, \epsilon]$ yielding 1

To prove 2 take $y \in P[x, \epsilon]$. Then $d(f^j(x), f^j(y)) \leq \epsilon$ for some $j \in \mathbb{Z}$. By taking $i = |j|$ we obtain $-i \leq j \leq i$ satisfying $d(f^j(x), f^j(y)) \leq \epsilon$. Hence $y \in P_i[x, \epsilon]$ for some $i \in \mathbb{N}$ thus proving $P[x, \epsilon] \subseteq \bigcup_{i \in \mathbb{N}} P_i[x, \epsilon]$. The reversed inclusion is obvious so (2) holds

To prove 3 it is already clear.

Lemma 2. If $f: X \rightarrow X$ is a homeomorphism of a compact metric space X , then ϕ_ϵ is a Borel map forevery $\epsilon > 0$.

Proof:

First note that $P_i[x, \epsilon] \subseteq P_{i'}[x, \epsilon]$ whenever $i \leq i'$. Then, $\phi_\epsilon = \sup_{i \in \mathbb{N}} \phi_{\epsilon, i}$ by Lemma 1.

It then suffices to show that $\phi_{\epsilon, i}$ is measurable for any $\epsilon > 0$ and $i \in \mathbb{N}$. For this we have to prove that the set $\phi_{\epsilon, i}^{-1}([-\infty, a])$ is a Borel set, $\forall a > 0$. Actually,

We shall prove that this set is open $\forall a > 0$. Take $x \in \phi_{\epsilon, i}^{-1}([-\infty, a])$ and a sequence $x_k \in X$ with $x_k \rightarrow x$. Because $x \in \phi_{\epsilon, i}^{-1}([-\infty, a])$, we have $\mu(P_i[x, \epsilon]) < a$.

As μ is regular [11], there is O open with $P_i[x, \epsilon] \subseteq O$ such that $\mu(O) < a$.

We claim that $P_i[x_k, \epsilon] \subseteq O$ for k large. Otherwise, by compactness, we can choose a sequence $y_k \in P_i[x_k, \epsilon] \setminus O$ such that $y_k \rightarrow y$ for some $y \in X$. As $y_k \notin O$ and O is open, we get $y \notin$

O. However, $y_k \in P_i[x_k, \epsilon]$ and then there is a sequence $-i \leq j_k \leq i$ satisfying $d(f^{j_k}(x_k), f^{j_k}(y_k)) \leq \epsilon$ for all $k \in \mathbb{N}$. Clearly we can assume $j_k = j$ for some fixed $-i \leq j \leq i$ hence $d(f^j(x_k), f^j(y_k)) \leq \epsilon$ for all $k \in \mathbb{N}$. Letting $k \rightarrow \infty$ above we obtain $d(f^j(x), f^j(y)) \leq \epsilon$. Since $-i \leq j \leq i$, this prove

$y \in P_i[x, \epsilon]$ and so $y \in O$. This is a contradiction which proves the claim. From this claim one has $\varphi_{\epsilon, i}(x_k) = \mu(P_{\epsilon, i}(x_k)) \leq \mu(O)$
 a , and so x_k belongs to $\varphi_{\epsilon, i}^{-1}([-\infty, a])$ for k
 large. Hence $\varphi_{\epsilon, i}^{-1}([-\infty, a])$ is open.

Lemma 3. If $f : X \rightarrow X$ is a homeomorphism of a compact metric space X , then $M_{\text{dis}}(f)$ is a Baire subset of $M(X)$.

Proof : For all $\delta, \epsilon > 0$ and $i \in \mathbb{N}$ we define
 $C(\epsilon, i, \delta) = \{\mu \in M(X) : \mu(P_i[x, \epsilon]) \geq \delta \text{ for some } x \in X\}$.

By **Lemma1** we have

$$\mu \in M(X) \setminus M_{\text{dis}}(f) \Leftrightarrow \mu(P(x)) > 0 \text{ for some } x \in X$$

$$\Leftrightarrow \exists n \in \mathbb{N}^+ \text{ such that } \lim_{m \rightarrow \infty} \mu(P[x, m^{-1}]) \geq n^{-1}$$

for some $x \in X$

$$\Leftrightarrow \exists n, m \in \mathbb{N}^+, x \in X \text{ such that } \mu(P[x, l^{-1}]) \geq n^{-1} \text{ for all } l \geq m$$

$$\Leftrightarrow \exists n, m \in \mathbb{N}^+ \text{ and } x \in X$$

such that

$$\forall l \geq m \exists i \in \mathbb{N}$$

such that

$$\mu(P_i[x, l^{-1}]) \geq n^{-1} \\ \Leftrightarrow \mu \in \bigcup_{n \in \mathbb{N}^+} \bigcup_{m \in \mathbb{N}^+} \bigcap_{l \geq m} \bigcup_{i \in \mathbb{N}} C(l^{-1}, i, n^{-1})$$

And so

$$M(X) \setminus M_{\text{dis}}(f) = \bigcup_{n \in \mathbb{N}^+} \bigcup_{m \in \mathbb{N}^+} \bigcap_{l \geq m} \bigcup_{i \in \mathbb{N}} C(l^{-1}, i, n^{-1}) \quad (2)$$

Next we prove that $C(\epsilon, i, \delta)$ is closed in $M(X)$ for any $\epsilon, \delta > 0$ and any $i \in \mathbb{N}$.

Fix $\epsilon, \delta > 0$ and $i \in \mathbb{N}$. Take a sequence $\mu_n \in C(\epsilon, i, \delta)$ converging to $\mu \in M(X)$ with respect to the weak* topology. As $\mu_n \in C(\epsilon, i, \delta)$ there is a sequence $x_n \in X$ with $x_n \rightarrow x$ such that $\mu(P_i[x_k, \epsilon]) \geq \delta$ for any $k \in \mathbb{N}$.

If we prove $\mu(P_i[x, \epsilon]) \geq \delta$ we would obtain $\mu \in C(\epsilon, i, \delta)$ and then $C(\epsilon, i, \delta)$ is closed in $M(X)$. Hence it suffices to prove $\mu(P_i[x, \epsilon]) \geq \delta$.

Take any compact neighborhood V of $P_i[x, \epsilon]$. We claim that $P_i[x_n, \epsilon] \subseteq \text{Int}(V)$ for k large. If not, there is a sequence $y_n \in P_i[x_n, \epsilon] \setminus \text{Int}(V)$ which, by compactness, we can further assume $y_k \rightarrow y$ for some $y \in X$.

On the one hand, we have $y \notin \text{Int}(V)$ because $\text{Int}(V)$ is open and, on the other, $y_n \in P_i[x_n, \epsilon]$ for any n

hence there is a sequence $-i \leq j_n \leq i$ satisfying

$$d(f^{j_n}(x_n), f^{j_n}(y_n)) \leq \epsilon, \forall n \in \mathbb{N}.$$

We can assume $j_n = j$ for some $-i \leq j \leq i$ hence

$$d(f^j(x_n), f^j(y_n)) \leq \epsilon, \forall n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ above we obtain

$$d(f^j(x), f^j(y)) \leq \epsilon,$$

and so $y \in P_i[x, \epsilon]$. Since V is a neighborhood of $P_i[x, \epsilon]$, we obtain $y \in \text{Int}(V)$ which is a contradiction. This contradiction shows $P_i[x_n, \epsilon] \subseteq \text{Int}(V)$ for k large and, then,

$$\delta \leq \mu_n(P_i[x_n, \epsilon]) \leq \mu_n(V), \forall n \text{ large.}$$

As $\mu_n \rightarrow \mu$ and V is closed, Theorem 6.1 in [11] implies

$$\delta \leq \limsup_{n \rightarrow \infty} \mu_n(V) = \mu(V).$$

Hence $\mu(V) \geq \delta$ for any compact neighborhood V of $P_i[x, \epsilon]$. From this we get

$$\mu(P_i[x, \epsilon]) \geq \delta$$

and so $C(\epsilon, i, \delta)$ is closed in $M(X)$, for all $\epsilon, \delta > 0$ and $i \in \mathbb{N}$.

Then, $\bigcup_{i \in \mathbb{N}} C(l^{-1}, i, n^{-1})$ is an F_σ subset of $M(X)$ ($\forall n, l \in \mathbb{N}^+$) from which we get that $\bigcap_{l \geq m} \bigcup_{i \in \mathbb{N}} C(l^{-1}, i, n^{-1})$ is an $F_{\sigma\delta}$ subset of $M(X)$ ($\forall m, n \in \mathbb{N}^+$). Since the space $M(X)$ of Borel probability measures equipped with the weak* topology is a compact (hence complete), every $F_{\sigma\delta}$ subset of it is Baire (cf. Lemma 1 in [8]). Consequently, $\bigcap_{l \geq m} \bigcup_{i \in \mathbb{N}} C(l^{-1}, i, n^{-1})$ is a Baire subset of $M(X)$, $\forall m, n \in \mathbb{N}^+$. Therefore, $M(X) \setminus M_{\text{dis}}(f)$ (and so $M_{\text{dis}}(f)$) are Baire subsets by (1). This finishes the proof

Lemma 4. Every almost distal measure of a homeomorphism $f : X \rightarrow X$ of a metric space X is nonatomic

Proof:

Suppose by contradiction that $\mu(\{x_0\}) > 0$ for some $x_0 \in X$. Since μ is almost distal, there is a measurable set E with $\mu(E) = 1$ such that $\mu(x) = 0$ for all $x \in E$. As $\mu(\{x_0\}) > 0$ and $\mu(E) = 1$, one has $x_0 \in E$ and so $\mu(\{x_0\}) = 0$ which is absurd. This completes the proof.

Lemma 5. If μ is an almost distal measure of a circle homeomorphism f , then $\text{supp}(\mu) \subseteq \Omega(f)$.

Proof:

We have that $S^1 \setminus \Omega(f)$ is a disjoint collection of open intervals J . It follows that $\text{diam}(f^n(J)) \rightarrow 0$ as $n \rightarrow \pm\infty$ hence $J \subseteq P(x)$ for every $x \in J$. Now, assume by contradiction that there is an almost distal measure μ with $\text{supp}(\mu) \not\subseteq \Omega(f)$. Then, $\mu(J) > 0$ for some interval J as above.

Hence $\mu(P(x)) \geq \mu(J) > 0$ for all $x \in J$ from which we get $J \subseteq S^1 \setminus \text{Dist}(f, \mu)$. It then follows that $\mu(S^1 \setminus \text{Dist}(f, \mu)) > 0$ and so $\mu(\text{Dist}(f, \mu)) < 1$ which is absurd.

Lemma 6. For every homeomorphism $f : X \rightarrow X$ of a metric space X and every $k \in \mathbb{N}^+$, every distal point of a Borel measure μ with respect to f is a distal point of μ with respect to f^k . In particular, every almost distal measure of f is an almost distal measure of f^k .

Proof:

Since $\inf_{n \in \mathbb{Z}} d(f^{kn}(x), f^{kn}(y)) = 0$ implies $\inf_{n \in \mathbb{Z}} d(f^n(x), f^n(y)) = 0$, one has $P_{f^k}(x) \subseteq P_f(x)$ and the proof follows.

Theorem: 4

A circle homeomorphism has an almost distal measure with full support if and only if it is distal. In particular, every countably-distal circle homeomorphism is distal. Every almost distal measure of a circle homeomorphism with rational rotation number is distal.

Proof: Let $f : S^1 \rightarrow S^1$ be a circle homeomorphism. First assume that f has an almost distal measure with full support. Then, $\Omega(f) = S^1$ (If μ is an almost distal measure of a circle homeomorphism f , then $\text{supp}(\mu) \subseteq \Omega(f)$.) and so f is topologically conjugated to a circle rotation. From this we obtain that f is distal and the proof follows. If f is countably distal, then f has a distal measure with full support (e.g. the Lebesgue measure) and so f is distal by the previous part.

Next assume that f has rational rotation number and, by contradiction, that f has an almost distal measure μ . Since f has rational rotation number, the periodic points of f have a common period $k \in \mathbb{N}^+$ (say). In addition, by using Lemma (For every homeomorphism $f : X \rightarrow X$ of a metric space X and every $k \in \mathbb{N}^+$, (For every homeomorphism $f : X \rightarrow X$ of a metric space X and every $k \in \mathbb{N}^+$ every distal point of a Borel measure μ with respect to f is a distal point of μ with respect to f^k . In particular, every almost distal measure of f is an almost distal measure of f^k .) and replacing f by f^k if necessary we can assume that $\Omega(f) = \text{Fix}(f) = \{x \in S^1 : f(x) = x\}$. Then, we have

$$P(x) = \begin{cases} \{x\}, & \text{if } x \in \text{Int}(\text{Fix}(f)); \\ \text{Cl}(J) & \text{if } J \text{ for some component } J \text{ of } S^1 \setminus \text{Fix}(f) \\ J \cup \{x\}, & x \in \text{Fr}(J) \text{ for some component } J \text{ of } S^1 \setminus \text{Fix}(f) \end{cases} \quad (3)$$

where $\text{Fr}(\cdot)$ and $\text{Cl}(\cdot)$ denote the boundary and closure operations in S^1 .

Suppose by contradiction that f has an almost distal measure μ which is not distal. Then, $\mu(P(x_0)) > 0$ for some $x_0 \in S^1$. Since μ is nonatomic by Lemma (Every almost distal measure of a homeomorphism $f : X \rightarrow X$ of a metric space X is nonatomic) by (3) applied to $x = x_0$ would imply $\mu(J) > 0$ for some connected component J of $S^1 \setminus \text{Fix}(f)$. However, $\text{supp}(\mu) \subseteq \Omega(f)$ (Lemma 5) so $\text{supp}(\mu) \subseteq \text{Fix}(f)$ thus $\mu(J) = 0$ since $J \subseteq S^1 \setminus \text{Fix}(f)$. This is a contradiction which proves the result.

Theorem: 6

Every homeomorphism with distal measures of a metric space has uncountably many almost periodic points. Every homeomorphism with almost distal measures of a compact metric space has infinitely many non-wandering points)

Let $f : X \rightarrow X$ is a homeomorphism of a metric space X . Assume that f has a distal measure μ . By Theorem 3, p.67 in [3] we have that for every $x \in X$ there is $x^* \in \text{AP}(f)$ such that $x^* \in P(x)$. On the other hand, since $x^* \in P(x)$, we have $x \in P(x^*)$ and so $X = \bigcup_{x^* \in \text{AP}(f)} P(x^*)$.

Now suppose by contradiction that $\text{AP}(f)$ is countable. Since the proximal cells are all Borel sets by Corollary (For every homeomorphism $f : X \rightarrow X$ of a metric space X and every $x \in X$ one has that $P(x)$ is an F_σ subset of X . In particular, $P(x)$ is a Borel set).

We would have $\mu(X) \leq \sum_{x^* \in \text{AP}(f)} \mu(P(x^*))$. But μ is distal so $\mu(P(x^*)) = 0$ for all $x^* \in \text{AP}(f)$ thus $\mu(X) = 0$ which is absurd. Therefore, $\text{AP}(f)$ is uncountable.

Now assume that X is compact and that f has an almost distal measure. If $\Omega(f)$ were finite, then the number of stable classes of f is finite. Applying Theorem 5. we obtain a contradiction. Therefore, $\Omega(f)$ is infinite.

References

- [1] A. Arbieto, C.A. Morales, Some properties of positive entropy maps, *Ergod. Theory Dyn. Syst.* 34 (3) (2014) 765–776.
- [2] A. Artigue, D. Carrasco-Olivera, A note on measure-expansive diffeomorphisms, *J. Math. Anal. Appl.* 428 (1) (2015) 713–716.
- [3] J. Auslander, Minimal flows and their extensions, in: *North-Holland Mathematics Studies*, vol. 153, in: *Notas de Matemática (Mathematical Notes)*, vol. 122, North-Holland Publishing Co., Amsterdam, 1988.
- [4] M. Brin, G. Stuck, *Introduction to Dynamical Systems*, Cambridge University Press, Cambridge, 2002.
- [5] B. Carvalho, W. Cordeiro, n -expansive homeomorphisms with the shadowing property, *J. Differ. Equ.* 261 (6) (2016) 3734–3755.
- [6] K. Kuratowski, *Topology*. Vol. II, new edition, revised and augmented. Translated from the French by A. Kirkor, Academic Press, New York, London; Państwowe Wydawnictwo Naukowe Polish Scientific Publishers, Warsaw, 1968.
- [7] J. Li, R. Zhang, Levels of generalized expansiveness, *J. Dyn. Differ. Equ.* (2017), <http://dx.doi.org/10.1007/s10884-015-9502-6>, in press.
- [8] C.A. Morales, On supports of expansive measures, preprint, arXiv:1601.03618 [math.DS].
- [9] C.A. Morales, A generalization of expansivity, *Discrete Contin. Dyn. Syst.* 32 (1) (2012) 293–301.
- [10] J. Palis, W. de Melo, *Geometric Theory of Dynamical Systems. An Introduction*, Springer-Verlag, New York, Berlin, 1982, translated from the Portuguese by A.K. Manning.
- [11] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, *Probab. Math. Stat.*, vol. 3, Academic Press, Inc., New York, London, 1967.
- [12] Z.L. Zhou, weakly almost periodic point and measure centre, *Sci. China Ser. A* 36 (2) (1993) 142–153.

