IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

Fuzzy Ideal Graphs In Semigroup Structures

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Abstract:

The primary objective of this study is to establish a connection between fuzzy theory, graph theory, fuzzy graph theory, and algebraic structures. To achieve this, we introduce several key concepts, including fuzzy graphs of semigroups, fuzzy ideal graphs of semigroups as a generalization of existing notions, and the concept of isomorphism of fuzzy graphs. Furthermore, we explore the properties of regular fuzzy graphs of semigroups.

Keywords: Semigroup, Fuzzy ideal, Graph, Fuzzy graph, Fuzzy semigroup, Isomorphism of fuzzy graphs of semigroups, Regular fuzzy graph of semigroup, Fuzzy ideal graph.

Introduction:

Semigroups are fundamental algebraic structures with applications in various engineering fields, including automata, formal languages, coding theory, and finite state machines. The formal study of semigroups began in the early 20th century. The introduction of fuzzy theory by Zadeh in 1965 revolutionized the field, enabling the development of theories addressing uncertainty. Atanassov's intuitionistic fuzzy sets and Rosenfeld's application of fuzzy sets to subgroup theory paved the way for further research. Subsequent studies by Kuroki, Jun, and Murali Krishna Rao explored fuzzy semigroups, semirings, and ideals. Mordeson et al. applied fuzzy semigroup theory to coding, finite state machines, automata, and formal languages. Meanwhile, graph theory, introduced by Euler in 1736, provides a convenient framework for representing relationships between objects. Graphs play a crucial role in modeling and analyzing complex systems.

The development of graph algorithms is a fundamental aspect of computer applications. A wide range of algorithms are utilized to solve problems that are modeled using graph structures. These algorithms enable the resolution of graph theoretical concepts, which are then applied to address corresponding problems in computer science applications. Furthermore, several computer programming languages provide built-in support for graph theory concepts, facilitating their implementation and utilization.

The concept of fuzzy graphs was pioneered by Kauffman in 1973, building on Zadeh's fuzzy relations. Rosenfeld's 1975 work further developed the theory, providing a framework for modeling real-life situations characterized by vagueness in object descriptions or relationships. Fuzzy graphs have since become a vital tool for representing relationships with uncertainty, serving as a generalization of Euler's graph theory. The field has undergone significant expansion, with researchers introducing notions like complement, bipolar fuzzy graphs, and intuitionistic fuzzy graphs. As a result, fuzzy graph theory has found increasing applications in modeling complex, real-time systems.

This paper presents a novel concept: the fuzzy ideal graph of a semigroup. This notion generalizes existing concepts, including fuzzy ideals of semigroups, intuitionistic fuzzy ideals of semigroups, fuzzy graphs, and graphs. We investigate and establish key properties of fuzzy ideal graphs.

Definition 1:A ordered pair (M, ·), where M is a non-empty set and "·" is an associative binary operation on M.

Definition 2:A subset T of a semigroup M is a subsemigroup if it satisfies the following conditions:

T is non-empty. T is closed under the semigroup operation, i.e., $T \times T \subseteq T$.

Definition 3: A non-empty subset T of a semigroup M is:

- A left ideal if $MT \subseteq T$.

- A right ideal if $TM \subseteq T$.

Definition 4:A non-empty subset T of a semigroup M is an ideal if it is both a left ideal ($MT \subseteq T$) and a right ideal ($TM \subseteq T$).

Definition 5: Let M be a non-empty set. A fuzzy subset of M is a mapping $f: M \to [0, 1]$ that assigns to each element x in M a value f(x) in the interval [0, 1].

Definition 6: Let M be a semigroup. A fuzzy subset μ of M is called a fuzzy subsemigroup of M if it satisfies the following condition: $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$.

Definition 7: Let M be a semigroup. A fuzzy subset μ of M is called a fuzzy left (right) ideal of M if it satisfies the following condition:

$\mu(xy) \geq \mu(y)$ for all $x, y \in M$ (fuzzy left ideal)

$\mu(xy) \geq \mu(x)$ for all $x, y \in M$ (fuzzy right ideal)

Definition 8: Let M be a semigroup. A fuzzy subset μ of M is called a fuzzy ideal of M if it satisfies the following condition:$\mu(xy) \geq \max\{\mu(x), \mu(y)\}$ for all $x, y \in M$.

Example: Let V = {a, b, c} and binary operation 0 · 0 on V is defined by

.	**a**	**b**	**c**
a	**a**	**a**	**c**
b	**a**	**b**	**c**
c	**c**	**c**	**c**

$\mu: V \to [0, 1]$ by μ(a) = 1/2, μ(b) = 1/4, μ(c) = 2/3. Then μ is a fuzzy subsemigroup and fuzzy ideal of semigroup

Definition 9:A graph is a pair (V, E), where V is a non-empty set and E is a set of unordered pairs of elements of V. The graph (V, E) is denoted by G(V, E).

Definition 10: The number of vertices in a graph G(V, E) is called an order of G(V, E) and it is denoted by |V |. For simplicity an edge {x, y} will be denoted by xy.

Definition 11: The number of edges in a graph G(V, E) is called a size of G(V, E) and it is denoted by |E|. Definition 12 Two vertices x and y in a graph G(V, E) are said to be adjacent or neighbors, if {x, y} is an edge of G(V, E).

Definition 12: The neighbor set of a vertex x of a graph G(V, E) is the set of all elements in V which are adjacent to x and it is denoted by N(x).

Definition 13:A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

FUZZY IDEAL GRAPH REPRESENTATION OF A SEMIGROUP

This section explores the concepts of fuzzy graphs, fuzzy ideal graphs, isomorphic fuzzy graphs, and regular fuzzy graphs in the context of semigroups. Unless otherwise stated, V represents a commutative semigroup.

Definition 14: Let G(V, E) be a graph and (V, ·) be a finite semigroup. Suppose μ is a fuzzy subset of V such that for any two vertices u and v connected by an edge in E, the membership value of their product uv is greater than or equal to the maximum of the membership values of u and v. Then, the graph G(V, E) is called a fuzzy graph of the semigroup V, denoted by G(V, E, μ).

Definition 15: Let G(V, E) be a complete graph, meaning that every vertex in V is connected to every other vertex. Then, a fuzzy graph of semigroup G(V, E, μ) is referred to as a fuzzy ideal graph of a semigroup V.

Definition 16: Let G(V, E, μ) be a fuzzy ideal graph. We define a fuzzy subset σ of V by $\sigma(x) = 1 - \mu(x)$ for all $x \in V$. This definition allows us to establish a connection between the fuzzy ideal graph and the theory of fuzzy graphs. Specifically, we show that the condition $\mu(uv) \geq \max\{\mu(u), \mu(v)\}$ for all $\{u, v\} \in E$ implies $\mu(uv) \leq \min\{\sigma(u), \sigma(v)\}$ for all $\{u, v\} \in E$. As a result, we obtain a fuzzy graph $G = (\mu, \sigma)$ in the sense of Rosenfeld. This construction provides a generalization of several important concepts, including fuzzy ideals of semigroups, intuitionistic fuzzy ideals of semigroups, fuzzy graphs, and graphs.

Definition 17: Let G(V, E, μ) be a fuzzy graph of a semigroup V. To define the order of this fuzzy graph, we sum up the membership values of all vertices in V. This sum is denoted by p and is calculated as $p = \sum_{x \in V} \mu(x)$.

Definition 18: Let G(V, E, μ) be a fuzzy graph of a semigroup V. To define the size of this fuzzy graph, we sum up the membership values of the products of adjacent vertices in E. This sum is denoted by q and is calculated as $q = \sum_{\{x,y\} \in E} \mu(xy)$.

Definition 19:Let G(V, E, μ) be a fuzzy graph of a semigroup V. To define the degree of a vertex v, we consider the adjacent vertices of v and sum up the membership values of the products of v with these adjacent vertices. This sum is denoted by D(v) and is calculated as $D(v) = \sum_{u \neq v \{u,v\} \in E} \mu(uv)$.

Theorem 1: Let G(V, E, μ) be a fuzzy ideal graph of a semigroup V with 0 element and 1 element. Then:(1) $\mu(0) \geq \mu(u)$, for all $u \in V$, (2) $\mu(0) \geq \mu(1)$

Proof. For any $v \in V$, we have $\mu(0) = \mu(0v) \geq \mu(v)$, by the definition of a fuzzy ideal graph. In particular, taking v = 1, we get $\mu(0) = \mu(0 \cdot 1) \geq \mu(1)$.

Example: Let G (V, E) be a graph with V = {a, b, c} and E = {(a, b),(b, c),(c, a)} and binary operation on V, "·" is defined by

.	a	b	c
a	a	a	c
b	a	b	c
c	c	c	c

Define μ: V → [0, 1] by μ(a) = 1/2, μ(b) = 1/4, μ(c) = 2/3. Here c is the 0 element and c are the 1 element. μ (0) ≥ μ(u), for all u ∈ V and μ (0) ≥ μ (1). Then μ is a fuzzy ideal of the semigroup V. Thus G (V, E, μ) is fuzzy ideal graph of the semigroup V. Order of fuzzy ideal graph = P v ∈V μ(v) = μ(a) + μ(b) + μ(c) = 17/12. Size of fuzzy ideal graph = P {u, v} ∈E μ (u v) = μ(ab)+μ (b c) +μ (c a) = 1/2 + 2/3 + 2/3 = 11/6.

Definition 20: Let G (V, E, μ) be a fuzzy graph of a semigroup V. A regular fuzzy graph is a fuzzy graph in which the degree of every vertex is the same. In other words, if D(v) = k for all v ∈ V, then G (V, E, μ) is called a regular fuzzy graph.

Definition 21: Let G (V, E, μ) be a fuzzy graph of a semigroup V. To define the total degree of a vertex u, we combine two measures: the degree of u, which represents the sum of the membership values of the products of u with its adjacent vertices, and the membership value of u itself. The total degree of u is denoted by TD(u) and is calculated as TD(u) = D(u) + μ(u).

Theorem 2: The size of a k-regular fuzzy graph G(V, E, μ) of a semigroup V is |V|k/2.

Proof: By definition, the size of G(V, E, μ) is ∑{u,v}∈E μ(uv). Since G(V, E, μ) is k-regular, we have ∑v∈V D(v) = 2∑{u,v}∈E μ(uv) = 2S(G). As D(v) = k for all v ∈ V, we have 2S(G) = ∑k = |V|k. Therefore, S(G) = |V|k/2.

Definition 22: If each vertex of G(V, E, μ) has the same total degree k, then fuzzy graph G(V, E, μ) is said to be totally regular fuzzy graph of total degree k.

Definition 23: Degree of vertex x of a graph G(V, E) is defined as the number of edges incident on x and it is denoted by d(x) or equivalently deg(x) = |N(x)|.

Theorem 3: Let G(V, E, μ) be a fuzzy graph with vertex set V = {v1, v2, · · · vn}. Then, the sum of the degrees of all vertices is greater than or equal to the sum of the products of the degrees and membership values of adjacent vertices. Mathematically, this can be expressed as:

∑vi∈V D(vi) ≥ ∑vi≠vj{vi,vj}∈E d(vi)μ(vj)

Proof: By definition of degree, D(vi) = $\sum v_i \neq v_j$,{vi,vj}∈E μ(vivj) ≥ ∑vi≠vj,{vi,vj}∈E d(vi)μ(vj). Summing over all vertices, we get the desired result.

Theorem 4: Let G(V, E, μ) be a fuzzy graph of a semigroup V. Then, the membership function μ is a constant function if and only if the following two conditions are equivalent:

(1) The fuzzy graph G(V, E, μ) is regular.

(2) The fuzzy graph G(V, E, μ) is totally regular.

Proof:(⇒) Suppose μ is a constant function and G(V, E, μ) is regular. Then, D(u) = k for all u ∈ V, and μ(u) = c for all u ∈ V. Thus, TD(u) = D(u) + μ(u) = k + c, for all u ∈ V, and G(V, E, μ) is totally regular.

(⇐) Suppose G(V, E, μ) is totally regular and TD(u) = k for all u ∈ V. Then, D(u) + μ(u) = k, for all u ∈ V. Since μ is a constant function, D(u) = k - c, for all u ∈ V, and G(V, E, μ) is regular.

Definition 24: Let G(V_1, E_1, μ_1) and G(V_2, E_2, μ_2) be fuzzy graphs of semigroups V_1 and V_2, respectively. To define an isomorphism between these two fuzzy graphs, we need to find a bijective map h: $V_1 \to V_2$ that satisfies three conditions:

(i) h is a semigroup isomorphism, meaning it preserves the semigroup operation.

(ii) h preserves the membership values, i.e., $\mu_1(x) = \mu_2(h(x))$ for all x ∈ V1.

(iii) h preserves the edge weights, i.e., $\mu_1(xy) = \mu_2(h(x)h(y))$ for all {x, y} ∈ E_1 and {h(x), h(y)} ∈ E_2.

If such a map h exists, we say that the two fuzzy graphs are isomorphic and write G(V_1, E_1, μ_1) ~= G(V_2, E_2, μ_2).

Theorem 5: Let G(V_1, E_1, μ_1) and G(V_2, E_2, μ_2) be isomorphic fuzzy graphs of semigroups V_1 and V_2, respectively. Then, their orders and sizes are equal.

Proof: Suppose G(V_1, E_1, μ_1) and G(V_2, E_2, μ_2) are isomorphic fuzzy graphs. Then, there exists an isomorphism h: $V_1 \to V_2$ such that $\mu_1(x) = \mu_2(h(x))$ and $\mu_1(xy) = \mu_2(h(x)h(y))$ for all {x, y} ∈ E_1 and {h(x), h(y)} ∈ E_2.

Thus, the order of G(V_1, E_1, μ_1) is $\sum v \in V_1\ \mu_1(v) = \sum v \in V_1\ \mu_2(h(v)) = \sum v \in V_2\ \mu_2(v)$, which is the order of G(V_2, E_2, μ_2).

Similarly, the size of G(V_1, E_1, μ_1) is $\sum x,y\} \in E_1\ \mu_1(xy) = \sum \{x,y\} \in E_1\ \mu_2(h(x)h(y)) = \sum \{h(x),h(y)\} \in E_2\ \mu_2(h(x)h(y))$, which is the size of G(V_2, E_2, μ_2).

Theorem 6: Let G(V_1, E_1, μ_1) and G(V_2, E_2, μ_2) be isomorphic fuzzy graphs. If G(V_1, E_1, μ_1) is a regular fuzzy graph of a semigroup V_1, then G(V_2, E_2, μ_2) is a regular fuzzy graph of a semigroup V_2.

Proof: Suppose h is the isomorphism of fuzzy graphs G(V_1, E_1, μ_1) and G(V_2, E_2, μ_2). Then, for all u ∈ V_1, we have:

$D(u) = \sum u \neq v, \{u,v\} \in E_1\ \mu_1(uv)$

$= \sum u \neq v, \{h(u),h(v)\} \in E_2\ \mu_2(h(u)h(v))$

= D(h(u)). This shows that G(V_2, E_2, μ_2) is also regular.

Theorem 7: Let G(V_1, E_1, μ_1) and G(V_2, E_2, μ_2) be isomorphic fuzzy graphs. If G(V_1, E_1, μ_1) is a totally regular fuzzy graph of semigroup then G(V_2, E_2, μ_2) is a totally regular fuzzy graph of semigroup.

Definition 25: Let G(V, E, μ) be a fuzzy graph of a semigroup V. The complement of G(V, E, μ) is denoted by G(V, E, μ^{-}) and is defined as follows:μ(xy) = μ(xy) - max{μ(x), μ(y)}, for all {x, y} ∈ E.

Theorem 8: Let G(V_1, E_1, μ_1) and G(V_2, E_2, μ_2) be fuzzy graphs of semigroups V_1 and V_2. To prove that these graphs are isomorphic if and only if their complements are isomorphic, we need to consider the definition of isomorphism and complement.

CONCLUSION:

This research introduces and examines several key concepts:

- Fuzzy graph of a semigroup: A generalization of fuzzy graphs and semigroups.
- Isomorphism of fuzzy graphs of semigroups: A notion of equivalence between fuzzy graphs of semigroups.
- Regular fuzzy graph of a semigroup: A concept that combines regularity with fuzzy graphs of semigroups.
- Fuzzy ideal graph of a semigroup: A generalization of fuzzy ideals and graphs.

We investigate the properties and relationships of these concepts, providing insights into their behavior and potential applications.

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